# Chapter 8: Second-order Circuits


**I. Introduction.**

**II. Finding initial and final values.**

**III. The source free series/parallel *RLC* circuits.**

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**V. General second-order circuits.**

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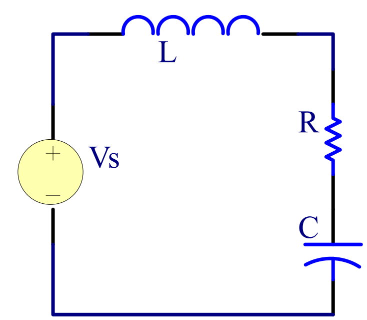
## I. Introduction

- In the previous chapter, we considered circuits with a single storage element (C or L). Such circuits are first-order because the differential equations describing them are first-order.
- In this chapter, we will consider circuits containing two storage elements (known as *second-order circuits*).
- A *second-order circuit* is characterized by a second-order differential equation. It consists of resistors and the equivalent of 02 energy storage elements.
- There are three kinds of second-order circuits:
  - Two storage elements of different type: *L* and *C*
  - Two storage elements of one type: *L* or *C*
  - An op amp circuit with 02 storage elements

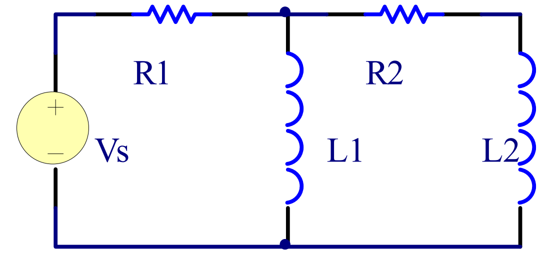



*Series RLC circuit*

# Chapter 8: Second-order circuits

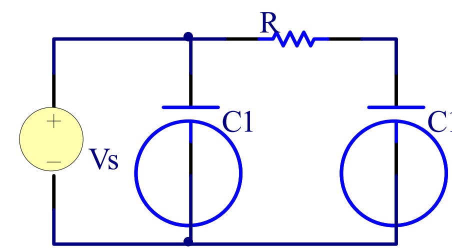
## I. Introduction

➢ The analysis of second-order will be similar to that used for first-order:



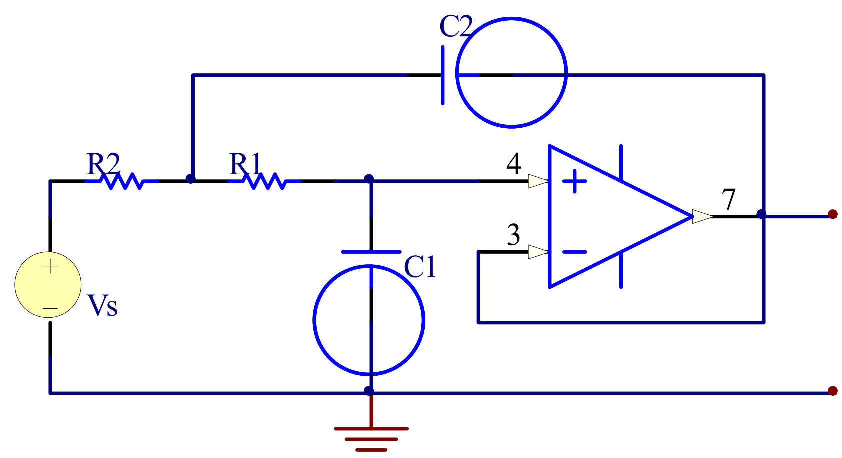

*RL circuit*

- ❖ First, consider circuits that are excited by the initial conditions of the storage elements → source free circuitss give natural responses.




*RC circuit*

- ❖ Second, with independent sources, circuits will give both the nature response and the forced response.




*Op amp with 2 storage element*

## II. Finding initial and final values

- The major problem in handing second-order circuits is finding the initial and final conditions on circuits variables:
  - Easy to get the initial and final values of $v$ and $i$
  - Difficulte to find the initial values of their derivatives: $dv/dt$, $di/dt$
- 02 key points in determining the initial conditions:
  - First, carefully handle the *polarity of voltage* $v_C(t)$, and the directions of $i_L(t)$
  - Second, keep in mind that $v_C(t)$, $i_L(t)$ *are always continuous*:

$$V_C(+0) = V_C(-0) \quad ; \quad i_L(+0) = i_L(-0)$$

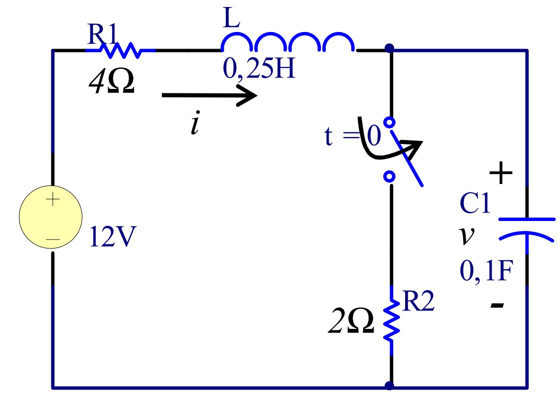
where:

*t = 0*: the time that the switching event takes place

*t = -0*: the time just before a switching event

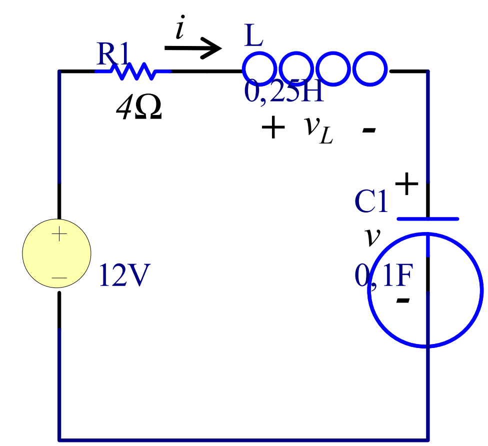
*t = +0*: the time just after a switching event

## II. Finding initial and final values




*Ex 8.1:* The switch has been closed for a long time, and opens at t = 0. Find $i(+0)$, $v(+0)$, $di(+0)/dt$, $dv(+0)/dt$, $i(\infty)$, $v(\infty)$

- t = -0: The circuit has reached DC steady state, so L acts like a short circuit, C acts like an open circuit

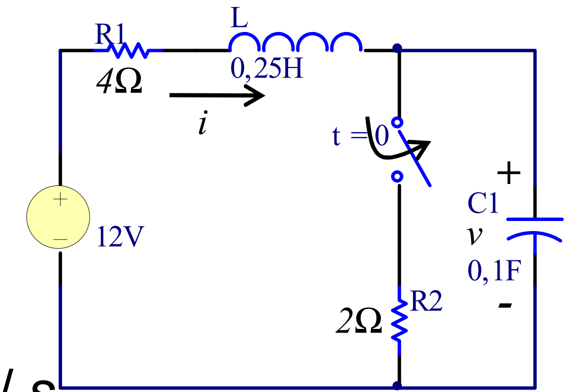



$$i(-0) = \frac{12}{R_1 + R_2} = 2A, \qquad v(-0) = 2.i(-0) = 4V$$

- As $v_C(t)$ and $i_L(t)$ cannot change abruptly

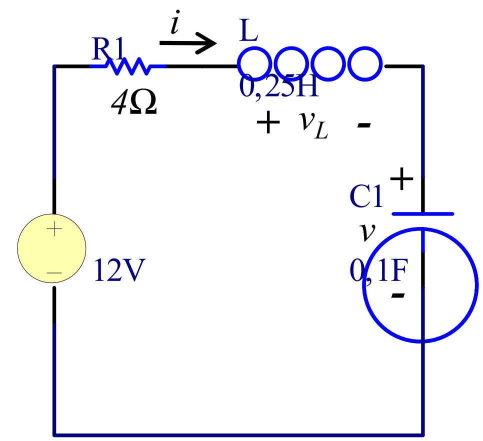
$$v_C(+0) = v_C(-0) = 4V \;\; ; \;\; i_L(+0) = i_L(-0) = 2A$$

### II. Finding initial and final values




*Ex 8.1:* The switch has been closed for a long time, and opens at t = 0. Find $i(+0)$, $v(+0)$, $di(+0)/dt$, $dv(+0)/dt$, $i(\infty)$, $v(\infty)$

- $t = +0$: $i_C = C\dfrac{dv_C}{dt} \rightarrow \dfrac{dv_C(+0)}{dt} = \dfrac{i_C(+0)}{C} = \dfrac{2}{0{,}1} = 20V/s$




- Applying KVL:

$$-12 + 4i(+0) + v_L(+0) + v_C(+0) = 0 \rightarrow v_L(+0) = 0$$

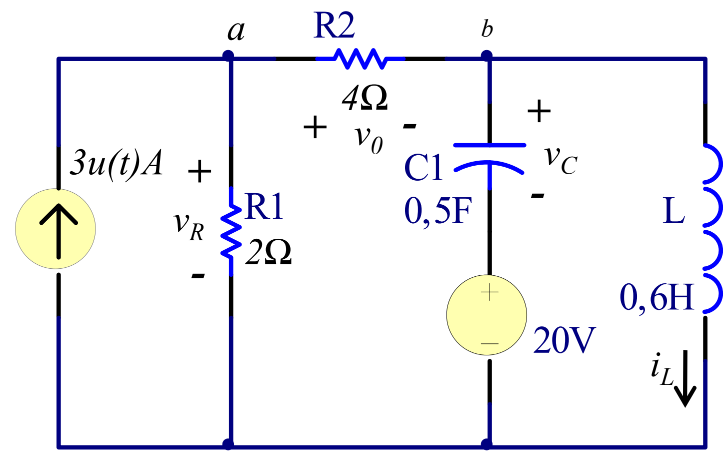
$$\rightarrow \frac{di(+0)}{dt} = \frac{v_L(+0)}{L} = 0A/s$$

- $t > 0$: the circuit undergoes transience

- $t > \infty$: Circuit reaches steady state again: $i(\infty) = 0A, \quad v(\infty) = 12V$

## II. Finding initial and final values

*Ex 8.2:* Find $i_L(+0)$, $v_C(+0)$, $v_R(+0)$, $di_L(+0)/dt$, $dv_C(+0)/dt$, $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$




- For $t = < 0$ : $3.u(t) = 0$
- At $t = -0$: Circuit has reached steady state

$$i_L(-0) = 0,\ v_R(-0) = 0,\ v_C(-0) = -20V$$

- For $t > 0$: $3.u(t) = 3$. Since $i_L$ and $u_C$ cannot change abruptly

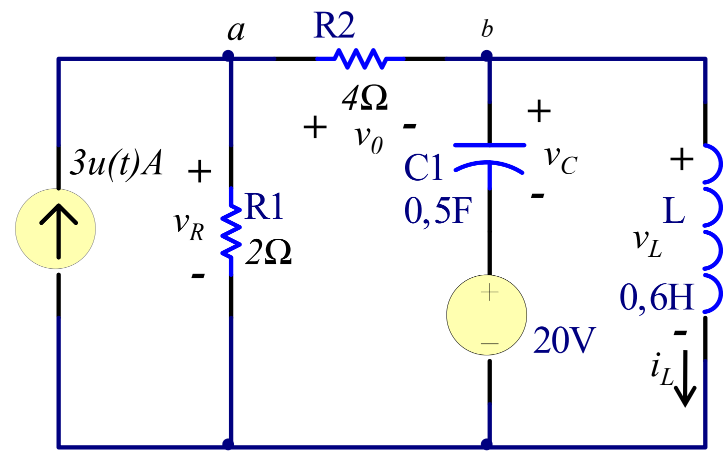
$$i_L(+0) = i_L(-0) = 0, \quad v_C(+0) = v_C(-0) = -20V$$

- Apply KCL at node *a:* $3 = \dfrac{v_R(+0)}{R_1} + \dfrac{v_0(+0)}{R_2} \longrightarrow v_R(+0) = v_0(+0) = 4V$

- Apply KVL to the middle mesh:

$$-v_R(+0) + v_0(+0) + v_C(+0) + 20 = 0 \xrightarrow{v_C(+0)=-20V} v_R(+0) = v_0(+0)$$

## II. Finding initial and final values




*Ex 8.2:* Find $i_L(+0)$, $v_C(+0)$, $v_R(+0)$, $di_L(+0)/dt$, $dv_C(+0)/dt$, $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$

- Since $L.di_L/dt = v_L$ $\rightarrow \dfrac{di_L(+0)}{dt} = \dfrac{v_L(+0)}{L}$
- Applying KVL in the right mesh gives:

$$v_L(+0) = v_C(+0) + 20 = -20 + 20 = 0 \rightarrow \frac{di_L(+0)}{dt} = 0$$

- Apply KCL at node *b:* $\dfrac{v_0(+0)}{R_2} = i_C(+0) + i_L(+0) \xrightarrow[i_L(+0)=0]{v_0(+0)=4} i_C(+0) = 1A$

$$\rightarrow \frac{dv_C(+0)}{dt} = \frac{i_C(+0)}{C} = \frac{1}{0,5} = 2V/s$$

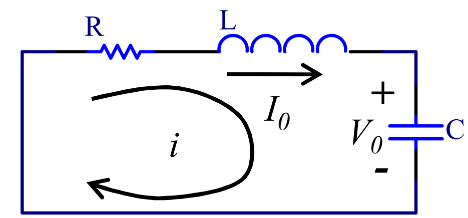
- Apply KCL to node a, and taking the derivative of each term and setting *t = +0*:

$$3 = \frac{v_R}{R_1} + \frac{v_0}{R_2} \leftrightarrow 0 = 2\frac{dv_R(+0)}{dt} + \frac{dv_0(+0)}{dt}$$

## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit

- An understanding of the natural response of the series RLC circuit is a necessary background for future studies in *filter design* and *communications networks*.
- Consider the series RLC circuit that is excited by the energy initially stored $V_0$, and $I_0$.

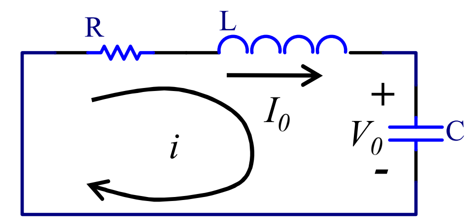



- At $t = 0$: $v(0) = \dfrac{1}{C}\displaystyle\int_{-\infty}^{0} i\,dt = V_0, \quad i(0) = I_0$
- Applying KVL to the loop: $Ri + L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int_{-\infty}^{t} i\,dt = 0 \leftrightarrow \boxed{\dfrac{d^2 i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{1}{LC} i = 0}$
- To solve a second-order differential equation it requires 02 initial conditions:
  - $i(+0)$ and $i'(+0)$, or
  - $i(+0)$ and $v(+0)$

## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit

- First, we find *i'(+0)*: $Ri(0) + L\dfrac{di(0)}{dt} + V_0 = 0 \rightarrow \dfrac{di(0)}{dt} = -\dfrac{1}{L}\left(RI_0 + V_0\right)$
- The solution of the second-order equation is: $i = A.e^{st}$
- Substituting the solution into the equation has:




$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

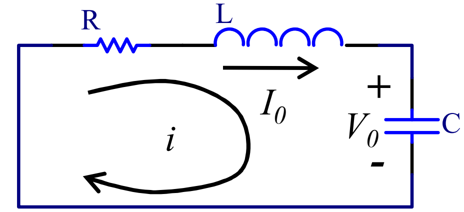
$$\rightarrow Ae^{st}\underbrace{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}_{characteristice\ equation} = 0 \rightarrow \begin{cases} s_1 = -\dfrac{R}{2L} + \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_1 = -\dfrac{R}{2L} - \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{cases}$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- roots $s_1$ and $s_2$ are called *natural frequencies* [Np/s]
- $_0$ : *resonant frequency* (*un-damped natural frequency, or damping factor*)

## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit




$$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{cases} \qquad \alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

- There are three types of solutions:
  - If $\alpha > \omega_0$: over damped case
  - *If* $\alpha = \omega_0$: critically damped case
  - *If* $\alpha < \omega_0$: under damped case

- 02 values of *s* show that there are 02 possible solutions for $i$

$$i_1 = A_1.e^{s_1 t}, \quad i_2 = A_2.e^{s_2 t} \rightarrow i(t) = A_1.e^{s_1 t} + A_2.e^{s_2 t}$$

$A_1$, $A_2$ are determined from the initial values $i(0)$ and $di(0)/dt$.

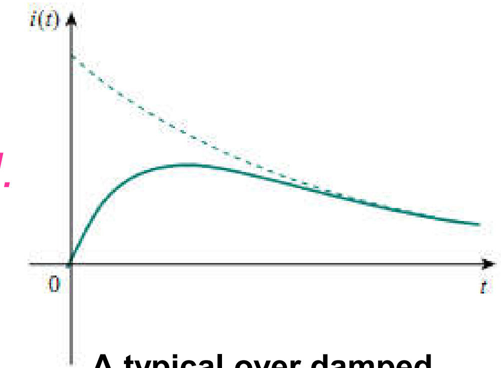
## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit

- *Over damped* case $\alpha > \omega_0 \rightarrow s_1$ and $s_2$ are *negative* and *real.*

$$\boxed{i(t) = A_1.e^{s_1 t} + A_2.e^{s_2 t}}$$

Response *decays* and *approaches* zero as *t* increases.

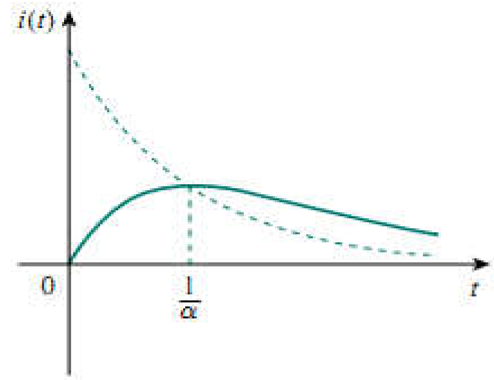



**A typical over damped response**

- *Critically damped* case $\alpha = \omega_0$

$$\boxed{i(t) = (A_1.t + A_2).e^{-\alpha t}}$$

Response reaches a maximum value at $t = 1/\alpha$, and then *decays* all the way to zero.




**A typical critically damped response**

## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit

➢ *Under damped* case $\alpha < \omega_0$:

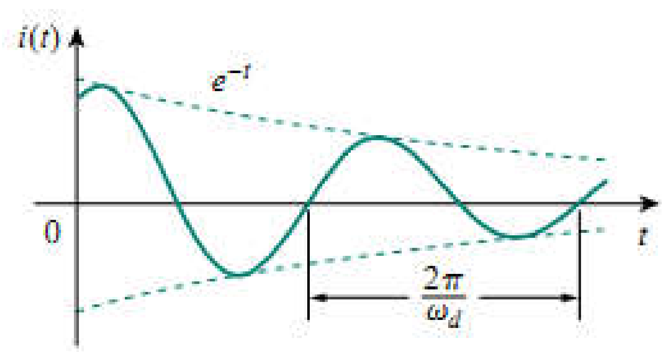
$$\rightarrow \begin{cases} s_1 = -\alpha + \sqrt{-\left(\omega_0^2 - \alpha^2\right)} = -\alpha + j\omega_d \\ s_1 = -\alpha - \sqrt{-\left(\omega_0^2 - \alpha^2\right)} = -\alpha - j\omega_d \end{cases}$$

$\omega_d$: damped natural frequency

$\omega_0$: un-damped natural frequencies

The natural response is:

$$\boxed{i(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)}$$




**A typical under damped response**

❖ The natural response is *exponentially damped* and *oscillatory* in nature.

❖ The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$

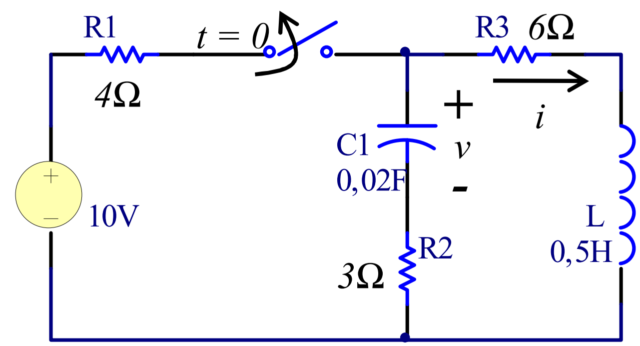
## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit

- Notes:
  - *Damping effect* is due to the presence of *R*
  - *Damping factor* $\alpha$ determines the rate at which the response is damped:
    - $\alpha = 0$: having LC circuit with $1/\sqrt{LC}$ as the un-damped natural frequency
    - $\alpha < \omega_0$: response is not only damped but also oscillatory.
  - By adjusting R, response may be made un-damped, over damped, critically damped, or under damped.
  - *Oscillatory response* is possible due to the presence of L, C: They allows the flow of energy back and forth.
  - The *critically damped* case is the *borderline* between the under damped and over damped cases.

## III. The source-free series/parallel *RLC* circuit

### III.1. The source-free series *RLC* circuit




*Ex 8.3:* Find $i(t)$ in the circuit. Assume that the circuit has reached steady state at $t = -0$.

- For $t < 0$: $i(0) = \dfrac{10}{R_1 + R_3} = 1A$ ; $v(0) = i.R_3 = 6V$

- For $t > 0$: Source – free series *RLC* circuit

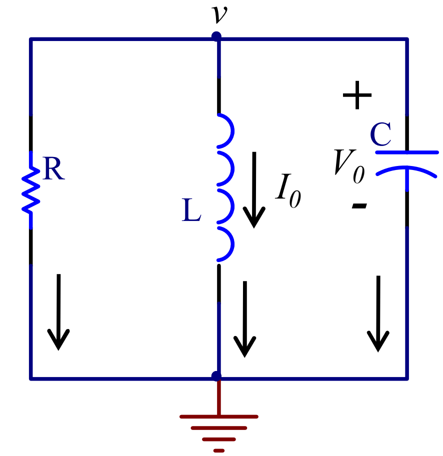
  - The roots:

$$\alpha = \frac{R_{eq}}{2L} = \frac{9}{2.0,5} = 9, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 10, \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm j4,359$$

  - The response is under damped: $i(t) = e^{-9t}\left(A_1 \cos 4,359t + A_2 \sin 4,359t\right)$

## III. The source-free series/parallel *RLC* circuit

### III.2. The source-free parallel *RLC* circuit




- Applying KCL: $\dfrac{v}{R}+\dfrac{1}{L}\int_{-\infty}^{t} v dt + C\dfrac{dv}{dt}=0 \rightarrow \dfrac{d^2v}{dt^2}+\dfrac{1}{RC}\dfrac{dv}{dt}+\dfrac{1}{LC}v=0$

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

$$\rightarrow s_{1,2}=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{RC}}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}$$

$$\omega_0=\frac{1}{\sqrt{LC}}$$

- Over damped $(\alpha>\omega_0)$: $\boxed{v(t)=A_1e^{s_1t}+A_2e^{s_2t}}$
- Critically damped $(\alpha=\omega_0)$: $\boxed{v(t)=(A_1+A_2t)e^{-\alpha t}}$
- Under damped $(\alpha<\omega_0)$: $s_{1,2}=-\alpha\pm j\omega_d,\quad \omega_d=\sqrt{\omega_0^2-\alpha^2}$

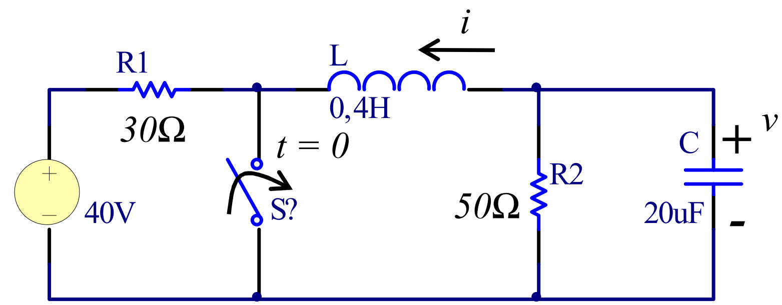
$$\boxed{v(t)=e^{-\alpha t}\left(A_1\cos\omega_d t+A_2\sin\omega_d t\right)}$$

$A_{1,2}$: determine from the initial conditions: $v(0)$, $dv(0)/dt$

## III. The source-free series/parallel *RLC* circuit

### III.2. The source-free parallel *RLC* circuit

*Ex 8.4:* Find v(t) for t > 0 in the RLC circuit




- When $t < 0$: The switch is opened.

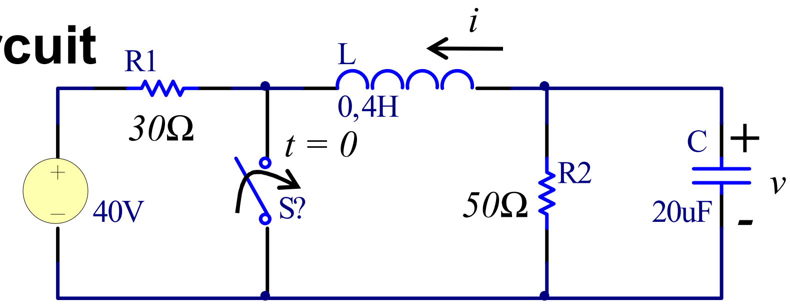
$$v(0)=\frac{R_2}{R_1+R_2}40=25V, \qquad i(0)=-\frac{40}{R_1+R_2}=-0,5A$$

- At $t = 0$: Applying the KCL gives:

$$\frac{v(0)}{R}+\frac{1}{L}\int_{-\infty}^{0} vdt + C\frac{dv(0)}{dt}=0 \rightarrow \frac{v(0)}{R}+i(0)+C\frac{dv(0)}{dt}=0$$

$$\rightarrow \frac{dv(0)}{dt}=-\frac{v(0)+Ri(0)}{RC}=-\frac{25-50.0,5}{50.20.10^{-6}}=0$$

## III. The source-free series/parallel *RLC* circuit




### III.2. The source-free parallel *RLC* circuit

*Ex 8.4:* Find v(t) for t > 0 in the RLC circuit

- When $t > 0$: The switch is closed. $\alpha = \dfrac{1}{2RC} = 500, \quad \omega_0 = \dfrac{1}{\sqrt{LC}} = 354$

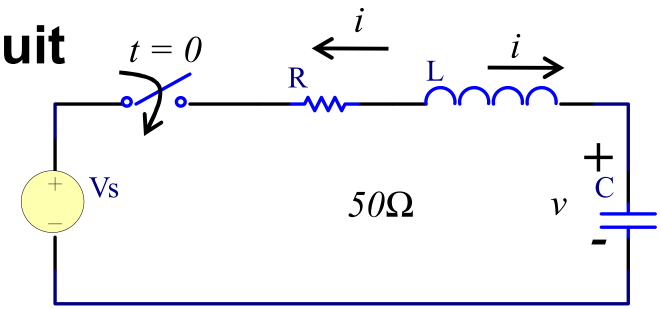
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \rightarrow \begin{cases} s_1 = -854 \\ s_2 = -146 \end{cases} \rightarrow v(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

At $t = 0$: $\begin{cases} v(0) = A_1 + A_2 = 25 \\ \dfrac{dv(0)}{dt} = -854A_1 - 146A_2 = 0 \end{cases} \rightarrow \begin{cases} A_1 = -5{,}16 \\ A_2 = 30{,}16 \end{cases}$

- The complete solution:

$$v(t) = -5{,}16e^{-854t} + 30{,}16e^{-146t}\,V$$

## IV. Step response of a series/parallel *RLC* circuit




### IV.1. Step response of a series *RLC* circuit

- For $t > 0$: Applying KVL around the loop

$$\begin{cases} L\dfrac{di}{dt} + Ri + v = V_S \\ i = C\dfrac{dv}{dt} \end{cases} \rightarrow \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

$$v(t) = v_n(t) + v_f(t)$$

$v_n(t)$: natural response

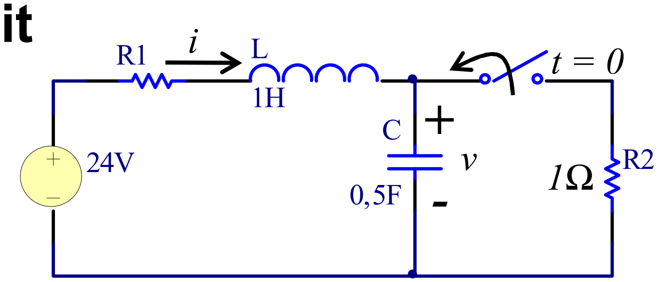
$v_S(t)$: forced response

- Over damped: $\boxed{v(t) = V_S + A_1e^{s_1t} + A_2e^{s_2t}}$
- Critically damped: $\boxed{v(t) = V_S + \left(A_1 + A_2t\right)e^{-\alpha t}}$
- Under damped: $\boxed{v(t) = V_S + \left(A_1\cos\omega_d t + A_2\sin\omega_d t\right)e^{-\alpha t}}$

$A_{1,\,2}$: determine from the initial conditions: $v(0)$, $dv(0)/dt$

## IV. Step response of a series/parallel *RLC* circuit

### IV.1. Step response of a series *RLC* circuit




*Ex 8.5:* Find $v(t)$ and $i(t)$ for $t > 0$ in the case of the different values of $R_1 = 5\Omega, 4\Omega, 1\Omega$

- ➢ **$R_1 = 5\Omega$**

  - ❖ For $t = 0$: $i(0) = \dfrac{24}{R_1 + R_2} = 4A, \quad v(0) = 1.i(0) = 4V \rightarrow \begin{cases} \alpha = \dfrac{R_1}{2L} = 2,5 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2 \end{cases}$

    $$\rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \rightarrow \begin{cases} s_1 = -1 \\ s_2 = -4 \end{cases}$$

  - ❖ For $t > 0$: $v(t) = v_f + A_1 e^{-t} + A_2 e^{-4t} = 24 + A_1 e^{-t} + A_2 e^{-4t}$

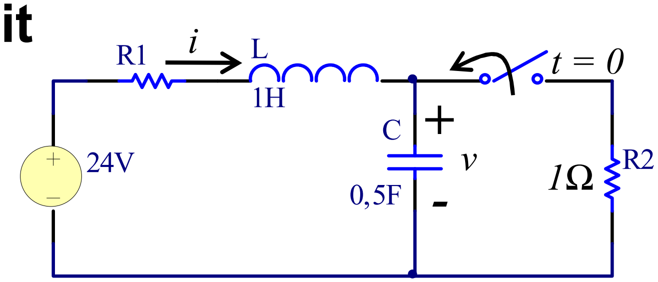
  - ❖ At $t = 0$: $v(0) = 24 + A_1 + A_2 = 4 \rightarrow A_1 + A_2 = -20$

    $$i(t) = C\frac{dv(t)}{dt} = C\left(-A_1 e^{-t} - 4A_2 e^{-4t}\right) \rightarrow i(0) = C\left(-A_1 - 4A_2\right) = 4$$

    $$\rightarrow v(t) = 24 + \frac{4}{3}\left(-16e^{-t} + e^{-4t}\right)V \qquad \rightarrow i(t) = C\frac{dv}{dt} = \frac{4}{3}\left(4e^{-t} - e^{-4t}\right)A$$

## IV. Step response of a series/parallel *RLC* circuit

### IV.1. Step response of a series *RLC* circuit




*Ex 8.5:* Find $v(t)$ and $i(t)$ for $t > 0$ in the case of the different values of $R_1 = 5\Omega, 4\Omega, 1\Omega$

- **$R_1 = 4\Omega$**
  - For $t = 0$: $i(0) = \dfrac{24}{R_1 + R_2} = 4,5A;\ v(0) = 1.i(0) = 4,5V \rightarrow \begin{cases} \alpha = \dfrac{R_1}{2L} = 2 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2 \end{cases}$
  - For $t > 0$: $v(t) = v_f + (A_1 + A_2 t)e^{-2t} = 24 + (A_1 + A_2 t)e^{-2t}$
  - At $t = 0$: $v(0) = 24 + A_1 = 4,5 \rightarrow A_1 = -19,5$

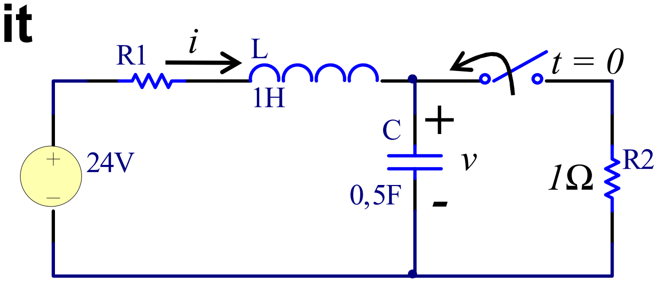
$$i(t) = C\frac{dv(t)}{dt} = C\left(-2A_1 - 2tA_2 + A_2\right)e^{-2t} \rightarrow i(0) = C\left(-2A_1 + A_2\right) = 4,5 \rightarrow A_2 = 57$$

$$\rightarrow v(t) = 24 + \left(-19,5 + 57t\right)e^{-2t}V \qquad \rightarrow i(t) = C\frac{dv}{dt} = \left(4,5 - 28,5t\right)e^{-2t}A$$

## IV. Step response of a series/parallel *RLC* circuit

### IV.1. Step response of a series *RLC* circuit

*Ex 8.5:* Find $v(t)$ and $i(t)$ for $t > 0$ in the case of the different values of $R_1$ = *5Ω, 4Ω, 1Ω*




- ***$R_1$ = 1Ω***
  - For $t = 0$: $i(0) = \dfrac{24}{R_1 + R_2} = 12A;\ v(0) = 1.i(0) = 12V \rightarrow \begin{cases} \alpha = \dfrac{R_1}{2L} = 0,5 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2 \end{cases}$

    $\rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0,5 \pm j1,936$

  - For $t > 0$: $v(t) = 24 + (A_1 \cos 1,936t + A_2 \sin 1,936t)e^{-0,5t}$
  - At $t = 0$: $v(0) = 24 + A_1 = 12 \rightarrow A_1 = -12$

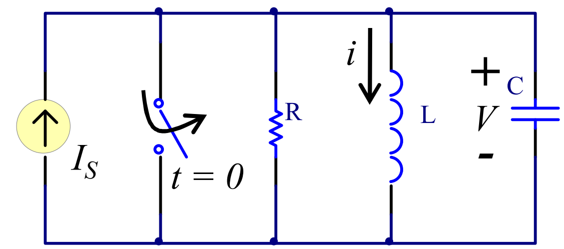
$$\frac{dv(t)}{dt} = e^{-0,5t}(-1,936A_1 \sin 1,936t + 1,936A_2 \cos 1,936t) - 0,5e^{-0,5t}\left(A_1 \cos 1,936t + A_2 \sin 1,936t\right)$$

$$\rightarrow \frac{dv(0)}{dt} = 1,936A_2 - 0,5A_1 = \frac{i(0)}{C} = 24 \rightarrow A_2 = 9,3$$

$$v(t) = 24 + (9,3\sin\omega_0 t - 12\cos\omega_{0t})e^{-0,5t}V \qquad i(t) = (18,582\sin\omega_0 t + 24\cos\omega_0 t)e^{-0,5t}A$$

## IV. Step response of a series/parallel *RLC* circuit

## IV.2. Step response of a parallel *RLC* circuit




- Applying KCL at the top node for t > 0

$$\frac{v}{R}+i+C\frac{dv}{dt}=I_S \xrightarrow{v=L\frac{di}{dt}} \frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{RC}=\frac{I_S}{LC}$$

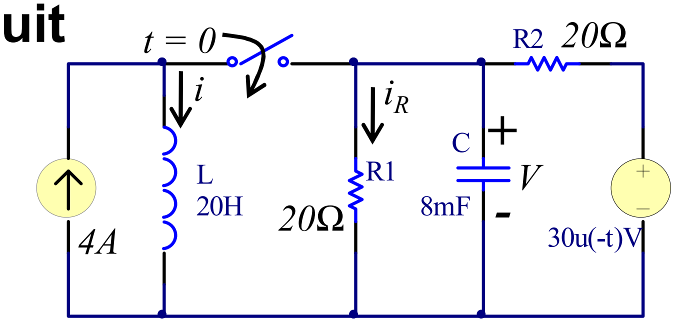
$$\rightarrow i(t)=i_n(t)+i_f(t) \qquad \begin{array}{l} i_n(t): \textit{natural response} \\ i_f(t): \textit{forced response} \end{array}$$

- Over damped: $\boxed{i(t)=I_S+A_1e^{s_1t}+A_2e^{s_2t}}$
- Critically damped: $\boxed{i(t)=I_S+\left(A_1+A_2t\right)e^{-\alpha t}}$
- Under damped: $\boxed{i(t)=I_S+\left(A_1\cos\omega_d t+A_2\sin\omega_d t\right)e^{-\alpha t}}$

$A_{1,\,2}$: determine from the initial conditions: i*(0), di(0)/dt*

## IV. Step response of a series/parallel *RLC* circuit

### IV.2. Step response of a parallel *RLC* circuit




*Ex 8.6:* Find $i(t)$, $i_R(t)$ for $t > 0$

- For $t < 0$: $\begin{cases} i(0) = 4A \\ v(0) = \dfrac{R_1}{R_1 + R_2} 30 = 15V \rightarrow \dfrac{di(0)}{dt} = \dfrac{v(0)}{L} = 0,75 \end{cases}$

- For $t > 0$: We have a parallel RCL circuit with current source $R = \dfrac{R_1 R_2}{R_1 + R_2} = 10\Omega$

$$\left.\begin{matrix} \alpha = \dfrac{1}{2RC} = 6,25 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2,5 \end{matrix}\right. \rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \rightarrow \begin{cases} s_1 = -11,978 \\ s_2 = -0,5218 \end{cases} \rightarrow i(t) = I_f + A_1 e^{-11,978t} + A_2 e^{-0,5218t}$$

- At $t = 0$: $\begin{cases} i(0) = 4 = 4 + A_1 + A_2 \rightarrow A_1 = -A_2 \\ \dfrac{di(0)}{dt} = -11,978A_1 - 0,5218A_2 = 0,75 \end{cases} \rightarrow \begin{cases} A_1 = -0.0655 \\ A_2 = 0.0655 \end{cases}$

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t})A \qquad i_R(t) = \frac{1}{20} L \frac{di}{dt} = 0,785e^{-11,978t} - 0,0342e^{-0,5218t} A$$

## V. General second order circuits

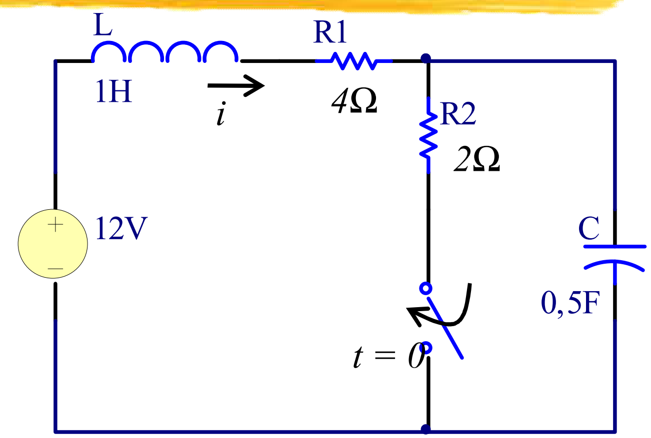
- Give a second order circuit, the step response $x(t)$ (current or voltage) can be determined by taking the following 5 steps:
  - Determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$
  - Find the natural response $x_n(t)$ (with 2 unknown constants) by turning off independent sources and applying KCL and KVL.
  - Obtain the forced response as: $x_f(t) = x(\infty)$
  - The total response is the sum of the natural response and forced response

$$x(t) = x_n(t) + x_f(t)$$

  - Determine the 2 unknown constants by imposing the initial conditions $x(0)$ and $dx(0)/dt$.

## V. General second order circuits

*Ex 8.7:* Find the complete response $v$ and $i$ for $t > 0$.




- Find the initial and final values:

$$\begin{cases} v(-0)=12V \\ i(-0)=0 \end{cases} \rightarrow \begin{cases} v(+0)=v(-0)=12V \\ i(+0)=i(-0)=0 \end{cases}$$

Applying KCL:

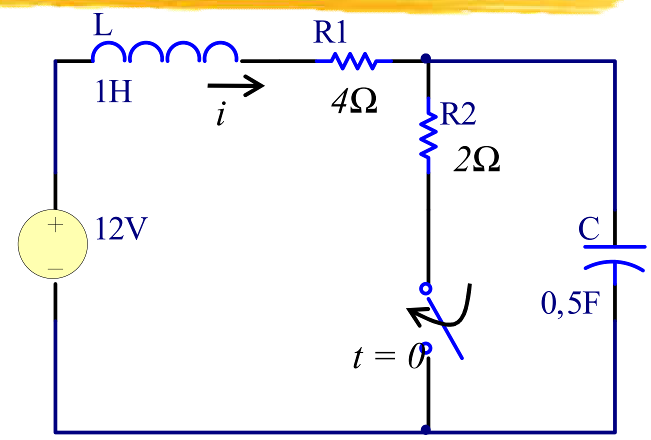
$$i(+0)=i_C(+0)+\frac{v(+0)}{R_2} \rightarrow i_C(+0)=-6A \rightarrow \frac{dv(+0)}{dt}=\frac{i_C(+0)}{C}=-12V/s$$

$$i(\infty)=\frac{12}{R_1+R_2}=2A, \quad v(\infty)=2.i(\infty)=4V=v_f(t)$$

- Find the natural response: Turn off the voltage source, and apply KCL, KVL

$$\begin{cases} i=\dfrac{v}{R_2}+C\dfrac{dv}{dt} \\ 4i+L\dfrac{di}{dt}+v=0 \end{cases} \rightarrow \begin{matrix} \dfrac{d^2v}{dt^2}+5\dfrac{dv}{dt}+6v=0 \\ s^2+5s+6=0 \end{matrix} \rightarrow v_n(t)=Ae^{-2t}+Be^{-3t}$$

## V. General second order circuits




*Ex 8.7:* Find the complete response $v$ and $i$ for $t > 0$.

- The complete response is:

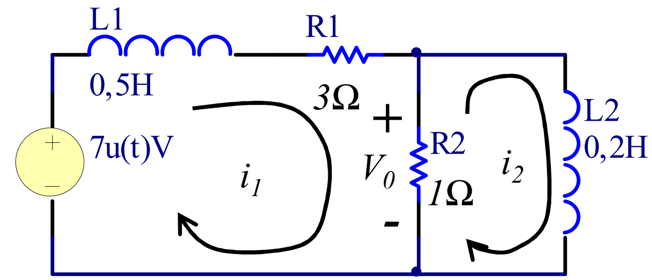
$$v(t) = v_n(t) + v_f(t) = 4 + A.e^{-2t} + B.e^{-3t}$$

- Imposing the initial condition gives:

$$\begin{cases} A + B = 8 \\ \dfrac{dv(0)}{dt} = -2A - 3B = -12 \end{cases} \rightarrow \begin{cases} A = 12 \\ B = -4 \end{cases} \quad \rightarrow v(t) = 4 + 12e^{-2t} - 4e^{-3t}\,V, \quad t > 0$$

$$i = \frac{v}{R_2} + C\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t} = 2 - 6e^{-2t} + 4e^{-3t}\,A, \quad t > 0$$

## V. General second order circuits




*Ex 8.8:* Find $v_0(t)$ for $t > 0$

- Obtain the initial and final values of 2 currents

$$\text{For } t<0:\ 7u(t)=0 \to i_1(-0)=0=i_2(-0)$$

$$\to \begin{array}{l} i_1(+0)=i_1(-0)=0, \quad i_2(+0)=i_2(-0)=0 \\ v_{L2}(+0)=v_0(+0)=R_2\left[i_1(+0)-i_2(+0)\right]=0 \end{array}$$

- Applying KVL to the left loop at $t = +0$

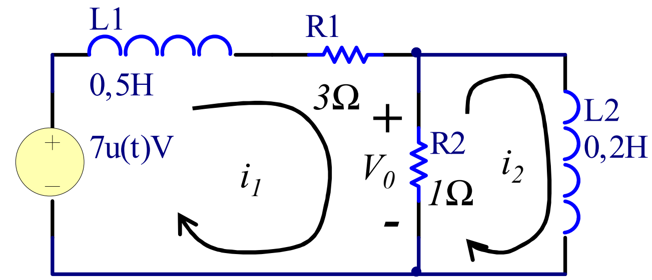
$$7=R_1 i_1(+0)+v_{L1}(+0)+v_0(+0)\to v_{L1}(+0)=7V \to \begin{array}{l} \dfrac{di_1(+0)}{dt}=\dfrac{v_{L1}}{L_1}=14V/s \\ \dfrac{di_2(t)}{dt}=\dfrac{v_{L2}}{L_2}=0 \end{array}$$

- As $t \to \infty$: $i_1(\infty)=i_2(\infty)=\dfrac{7}{R_1}=2,33A$

- Applying KVL to 2 meshes to find natural responses:

$$\begin{cases} (R_1+R_2)i_1-R_2 i_2+L_1\dfrac{di_1}{dt}=0 \\ R_2(i_2-i_1)+L_2\dfrac{di_2}{dt}=0 \end{cases} \to \begin{array}{l} \dfrac{d^2 i_1}{dt^2}+13\dfrac{di_1}{dt}+30i_1=0 \\ s^2+13s+30=0 \end{array}$$

## V. General second order circuits




*Ex 8.8:* Find $v_0(t)$ for $t > 0$

$$s^2+13s+30=0 \rightarrow \begin{cases} s_1=-3 \\ s_2=-10 \end{cases} \rightarrow i_{1n}=Ae^{-3t}+Be^{-10t}$$

$$\rightarrow i_1(t)=i_{1f}+i_{1n}=2,33+Ae^{-3t}+Be^{-10t}$$

- Imposing the initial condition gives:

$$\begin{cases} A+B+2,33=0 \\ -3A-10B=14 \end{cases} \rightarrow \begin{cases} A=-1,33 \\ B=-1 \end{cases} \rightarrow i_1(t)=2,33-1,33e^{-3t}-e^{-10t}A$$

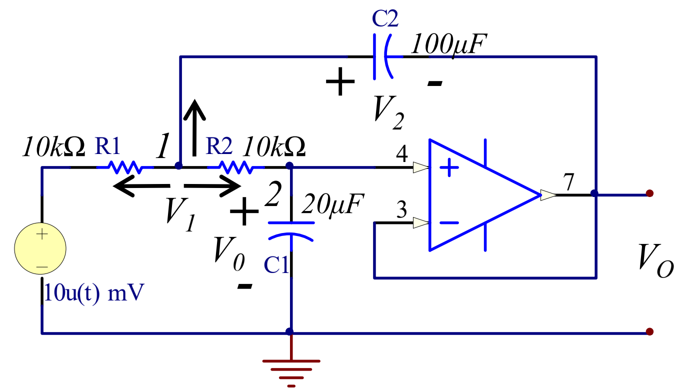
- Applying KVL to loop

$$7=4i_1-i_2+L_1\frac{di_1}{dt} \rightarrow i_2=-7+4i_1+L_1\frac{di_1}{dt}=2,33-3,33e^{-3t}+e^{-10t}A$$

$$\rightarrow v_0(t)=R_2\left[i_1(t)-i_2(t)\right]=2\left(e^{-3t}-e^{-10t}\right)A$$

## V. General second order circuits

*Ex 8.9:* Find $v_0(t)$ for $t > 0$




- Applying KCL:

$$At\ node\ 1: \frac{v_S - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_0}{R_2}$$

$$At\ node\ 2: \frac{v_1 - v_0}{R_2} = C_1\frac{dv_0}{dt} \quad \rightarrow v_1 = v_0 + R_2C_1\frac{dv_0}{dt}$$

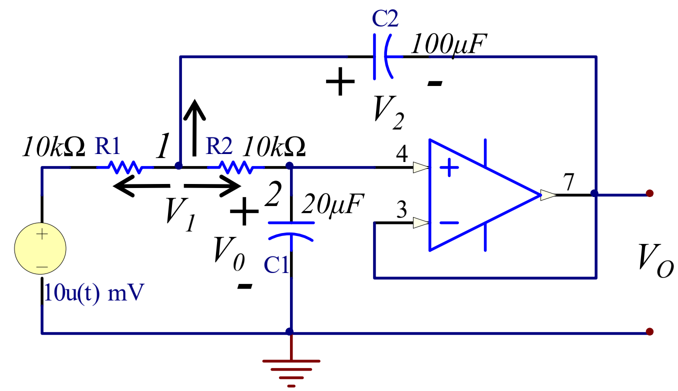
$$\xrightarrow{v_2 = v_1 - v_0} \frac{v_S - v_1}{R_1} = C_2\frac{dv_2}{dt} + C_1\frac{dv_0}{dt}$$

$$\rightarrow \frac{d^2v_0}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_0}{dt} + \frac{v_0}{R_1R_2C_1C_2} = \frac{v_S}{R_1R_2C_1C_2} \quad \rightarrow \frac{d^2v_0}{dt^2} + 2\frac{dv_0}{dt} + 5v_0 = 5v_S$$

- For the natural response, turning off the source: $s^2 + 2s + 5 = 0 \rightarrow s_{1,2} = -1 \pm j2$

$$\rightarrow v_{0n}(t) = e^{-t}\left(A\cos 2t + B\sin 2t\right)$$

## V. General second order circuits




*Ex 8.9:* Find $v_0(t)$ for $t > 0$

- As $t \to \infty$: $v_0(\infty) = v_1(\infty) = v_S \to v_{0f} = v_0(\infty) = 10mV$
- The complete response is:

$$\to v_0(t) = v_{0n} + v_{0f} = 10 + e^{-t}\left(A\cos 2t + B\sin 2t\right)$$

- Find initial conditions:

$$v_0(-0) = v_2(-0) = 0 \to \begin{matrix} v_0(+0) = v_2(+0) = 0 \\ v_1(+0) = v_2(+0) + v_0(+0) = 0 \end{matrix} \to \frac{dv_0(+0)}{dt} = \frac{v_1 - v_0}{R_2C_1} = 0$$

$$\to \begin{cases} v_0(+0) = 10 + A = 0 \to A = -10 \\ \dfrac{dv_0(+0)}{dt} = -A + 2B = 0 \to B = -5 \end{cases}$$

- The complete response becomes:

$$\to v_0(t) = 10 - e^{-t}\left(10\cos 2t + 5\sin 2t\right) mV$$

## VI. Applications

- Practical applications of *RLC* circuits are found in control and communications circuits, for examples:
  - Ringing circuits
  - Peaking circuits
  - Resonant circuits
  - *Smoothing circuits*
  - Filters
  - *Automobile ignition*
- Most of the circuits cannot be covered until we treat AC sources.