Chapter 7: First-order Circuits

I. Introduction.

II. The source-free RC/RL circuit.

III. Singularity functions.

VI. Step response of an RC/RL circuit.

V. First-order op amp circuits

VI. Applications.

I. Introduction

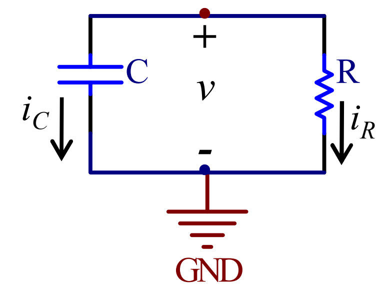
- Have considered: passive elements (R, L, C) and active element (OPAM).
- This chapter will study 2 types of simple circuits: R-C, R-L
- Applying KCL, KVL in these R-C, R-L circuits gives first-order differential equations.
- A *first-order* circuit is characterized by a first-order differential equation.
- There are 2 ways to excite the circuits:
  - *Source-free circuit:* Energy is initially stored in the capacitive or inductive element. The energy (*dependent source*) causes current to flow in the circuit and is dissipated in the resistors.
  - *Independent source: dc source (sinusoidal source, exponential source).*
- Applications: delay/relay circuit, photoflash unit, automobile ignition circuit.

II. The source-free RC/RL circuit

II.1. R-C circuit

➢ A *source-free R-C circuit* occurs when its dc source is suddenly disconnected, the energy already stored in the capacitor is released to the resistors.

➢ Consider a series combination of a resistor and an initially charged capacitor with V_0 → determine the circuit response.



❖ Value of energy stored in the capacitor: $\omega = \frac{1}{2}CV_0^2$

❖ Applying KCL at the top node of the circuit:

$$i_C + i_R = 0 \to C\frac{dv}{dt} + \frac{v}{R} = 0 \to \frac{dv}{dt} + \frac{v}{RC} = 0$$

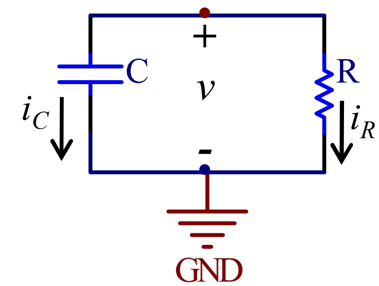
(First-order differential equation)

$$\leftrightarrow \frac{dv}{v} = -\frac{1}{RC}dt \to \ln v = -\frac{t}{RC} + \ln A \to \ln\frac{v}{A} = -\frac{t}{RC} \to v(t) = A.e^{-\frac{t}{RC}}$$

$$\xrightarrow{v(0)=A=V_0} \boxed{v(t) = V_0.e^{-\frac{t}{RC}}}$$

II. The source-free RC/RL circuit

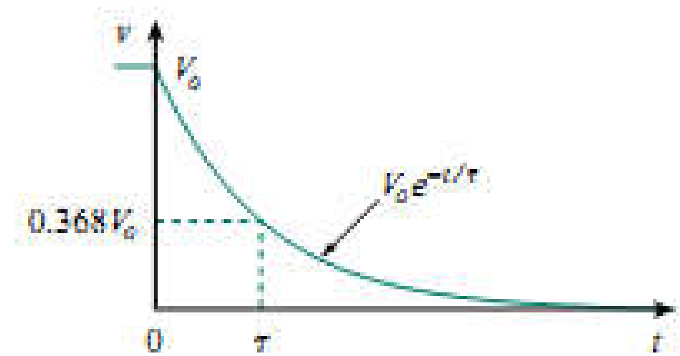
II.1. R-C circuit



$$v(t) = V_0.e^{-\frac{t}{RC}}$$

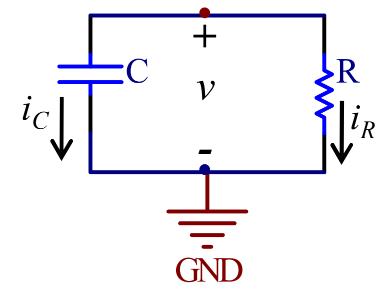
- Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natured response* of the circuit.
- The *natural response* of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
- The *time constant* of a circuit is the time required for the response to decay by a factor of 1/e or 36,8% of its initial value.

$$\tau = RC \quad \rightarrow \quad v(t) = V_0 e^{-\frac{t}{\tau}}$$



II. The source-free RC/RL circuit

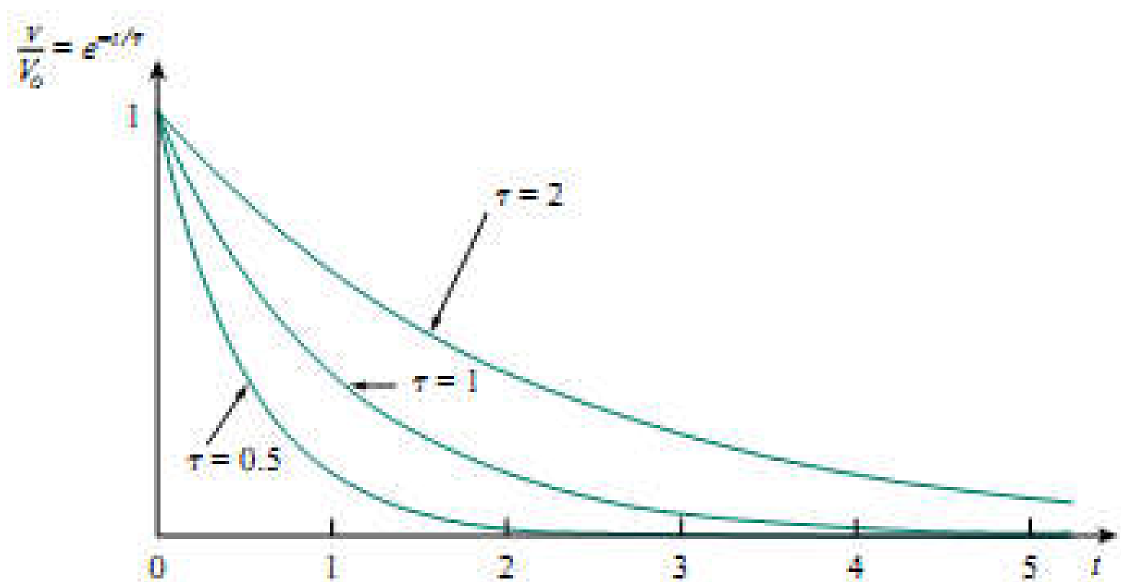
II.1. R-C circuit



- After 5τ**,** the voltage *v(t)* is less than *1%* of V_0 → the capacitor is fully discharged (or charged).
- In other words, after 5τ for the circuit to reach its *final state (steady state)* when no changes take place with time.

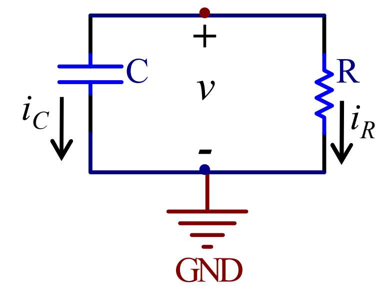
$$v(t) = V_0 e^{-\frac{t}{\tau}}$$

t	$v(t) / V_0$
τ	0.36788
2τ	0,13534
3τ	0,04979
4τ	0,01832
5τ	0,00674



II. The source-free RC/RL circuit

II.1. R-C circuit



- The current $i_R(t)$: $i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}}$

- The power dissipated in the resistor: $p(t) = v.i_R = \frac{V_0^2}{R} e^{-\frac{2t}{\tau}}$

- The energy absorbed by the resistor up to time t is:

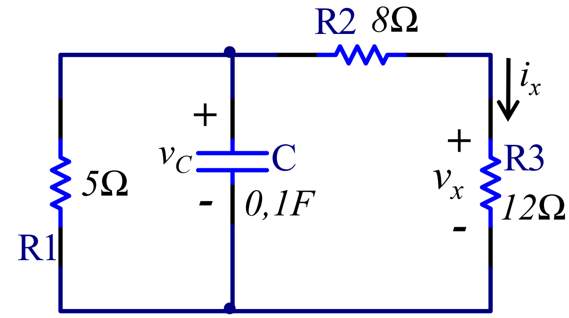
$$\omega_R(t) = \int_0^t p dt = \int_0^t \frac{V_0^2}{R} e^{-\frac{2t}{\tau}} dt = -\frac{\tau V_0^2}{2R} e^{-\frac{2t}{\tau}} \Bigg|_0^t = \frac{1}{2} C V_0^2 \left(1 - e^{-\frac{2t}{\tau}}\right)$$

- Note that: $t \to \infty, \; then \; \omega_R(\infty) = \frac{1}{2} C V_0^2 = \omega_C(0)$ *(the energy initially stored in the capacitor)*

II. The source-free RC/RL circuit

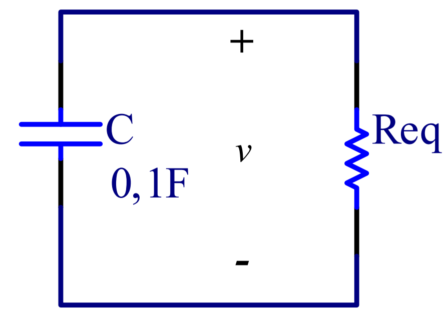
II.1. R-C circuit

Ex 7.1: Find v_C, v_x and i_x for $t > 0$ if $V(0) = 15V$



- The equivalent resistance:

$$R_{eq} = (R_2 + R_3)//R_1 = \frac{(12+8).5}{12+8+5} = 4\Omega$$



- The time constant: $\tau = C.R_{eq} = 40.0,1 = 0,4s$

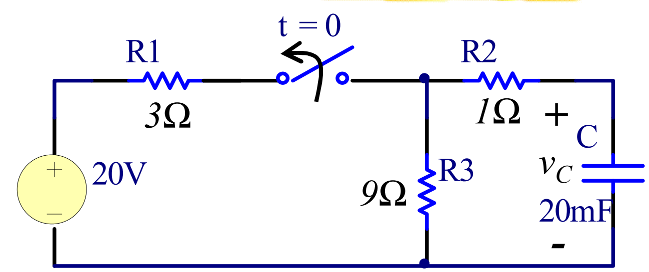
- Thus: $v_C(t) = v(0).e^{-\frac{t}{\tau}} = 15.e^{-\frac{t}{0,4}}V = 15.e^{-2,5t}V$

$$v_x(t) = \frac{R_3}{R_2 + R_3} v_C(t) = \frac{12}{12+8} 15.e^{-2,5t} = 9.e^{-2,5t}V$$

$$\rightarrow i_x = \frac{v_x(t)}{R_3} = \frac{9.e^{-2,5t}}{12} = 0,75e^{-2,5t}A$$

II. The source-free RC/RL circuit

II.1. R-C circuit



Ex 7.2: The switch has been closed for a long time, it is opened at $t = 0$. Find $v_C(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

- For $t < 0$, the switch is closed → C is open circuit to DC.

$$v_C(t) = \frac{R_3}{R_1 + R_3}.20 = 15V \quad with\ t<0 \quad \rightarrow \quad v_C(0) = V_0 = 15V$$

- For $t > 0$: Source free R-C circuit: $\begin{cases} R_{eq} = R_2 + R_3 = 10\Omega \\ V_0 = 15V \end{cases} \rightarrow \tau = R_{eq}.C = 0,2s$

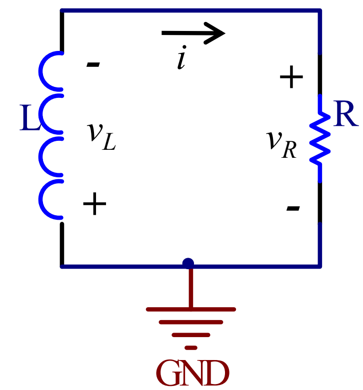
- The voltage across the capacitor for $t \geq 0$: $v_C(t) = V_0.e^{-\frac{t}{\tau}} = 15.e^{-5t}\,V$

- The initial energy stored in the capacitor:

$$\omega_C(0) = \frac{1}{2}C.v_C^2(0) = \frac{1}{2}.20.10^{-3}.15^2 = 2,25J$$

II. The source-free RC/RL circuit

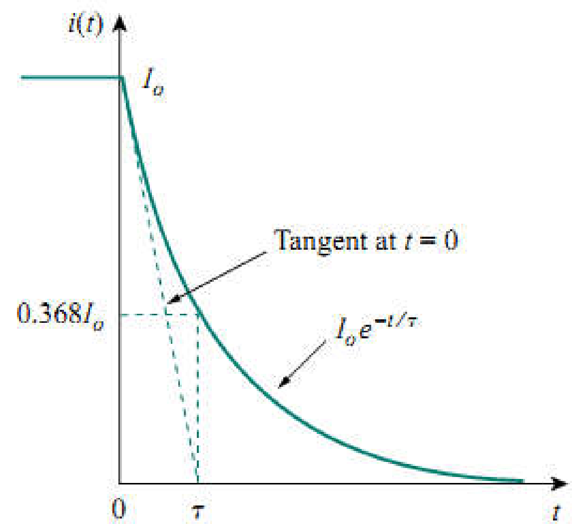
II.2. R-L circuit



- Consider the *R-L* circuit: $i_L(0) = I_0$, *initial current*
- The corresponding energy stored in the inductor as: $\omega(0) = \dfrac{1}{2}LI_0^2$
- Applying KVL to the loop:

$$v_L(t) + v_R(t) = 0 \leftrightarrow L\frac{di}{dt} + Ri = 0 \rightarrow \frac{di}{dt} + \frac{R}{L}i = 0 \rightarrow \int_{I_o}^{i(t)} \frac{di}{i} = -L\int_0^t \frac{R}{L}dt$$

$$\rightarrow \boxed{i(t) = I_0.e^{-\frac{R}{L}t} \;\; ; \;\; \tau = \frac{L}{R}}$$



- The energy absorbed by the resistor

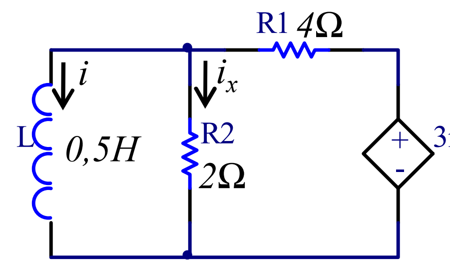
$$\omega_R(t) = \int_0^t pdt = \int_0^t I_0^2 Re^{-2t/\tau}dt = \frac{1}{2}LI_0^2\left(1 - e^{-2t/\tau}\right)$$

$$When\ t \rightarrow \infty: \;\; \omega_R(\infty) \rightarrow \frac{1}{2}LI_0^2 = \omega_L(0)$$

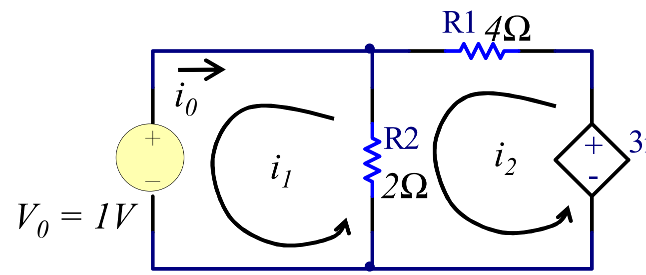
II. The source-free RC/RL circuit

II.2. R-L circuit

Ex 7.3: Assuming that $i(0) = 10A$, calculate $i(t)$ and $i_x(t)$



- *Solution 1:* Obtain the equivalent resistance



  - Applying KVL to the two loops results:

$$\begin{cases} R_2(i_1 - i_2) + V_0 = 0 \\ (R_1 + R_2)i_2 - R_2 i_1 - 3i_1 = 0 \end{cases} \rightarrow \begin{cases} i_1 - i_2 = -\dfrac{1}{2} \\ i_2 = \dfrac{5}{6} i_1 \end{cases} \rightarrow \begin{cases} i_1 = -3A \\ i_0 = -i_1 = 3A \end{cases}$$

  - Hence: $R_{eq} = R_{Th} = \dfrac{v_0}{i_0} = \dfrac{1}{3}\Omega$ The time constant is: $\tau = \dfrac{L}{R_{eq}} = \dfrac{1/2}{1/3} = 1,5s$

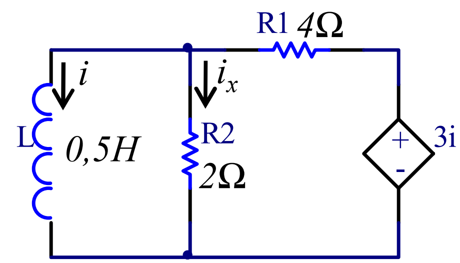
  - Thus, the current through the inductor is:

$$i(t) = i(0).e^{-t/\tau} = 10.e^{-2/3t} A \ , \ t > 0$$

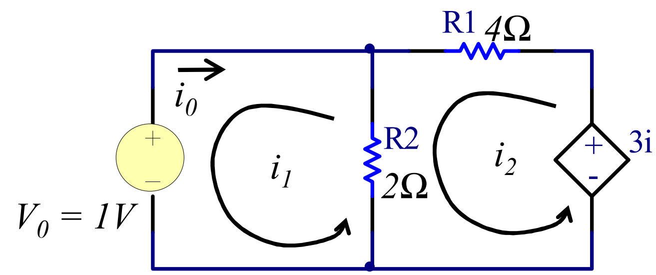
II. The source-free RC/RL circuit

II.2. R-L circuit

Ex 7.3: Assuming that $i(0) = 10A$, calculate $i(t)$ and $i_x(t)$



- *Solution 2:* Applying directly KVL to the circuit



$$\begin{cases} L\dfrac{di_1}{dt} + R_2(i_1 - i_2) = 0 \\ (R_1 + R_2)i_2 - R_2 i_1 - 3i_1 = 0 \end{cases} \rightarrow \begin{cases} \dfrac{di_1}{dt} + 4i_1 - 4i_2 = 0 \\ i_2 = \dfrac{5}{6} i_1 \end{cases}$$

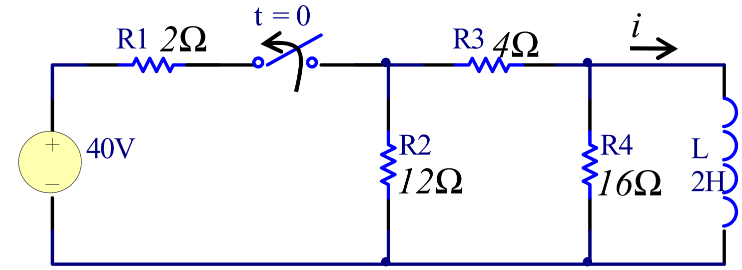
$$\rightarrow \frac{di_1}{dt} + \frac{2}{3} i_1 = 0 \rightarrow i_1(t) = i(0).e^{-2/3t} = 10e^{-2/3t} A \ , \ t > 0$$

- The voltage across the inductor: $v = L\dfrac{di}{dt} = -\dfrac{0,5.10.2}{3} e^{-2/3t} = -\dfrac{10}{3} e^{-2/3t} V$

- The current flows though R_2: $i_x(t) = \dfrac{v}{2} = -1,67.e^{-2/3t} A \ , \ t > 0$

II. The source-free RC/RL circuit

II.2. R-L circuit



Ex 7.4: The switch has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

- t < 0: the inductor acts as a short circuit

$$i_{R1}(t) = \frac{40}{R_1 + \dfrac{R_2R_3}{R_2R_3}} = 8A \rightarrow i(t) = \frac{R_2}{R_2 + R_3} i_{R1}(t) = 6A \rightarrow i(0) = i(-0) = 6A$$

- t > 0: *R-L* circuit

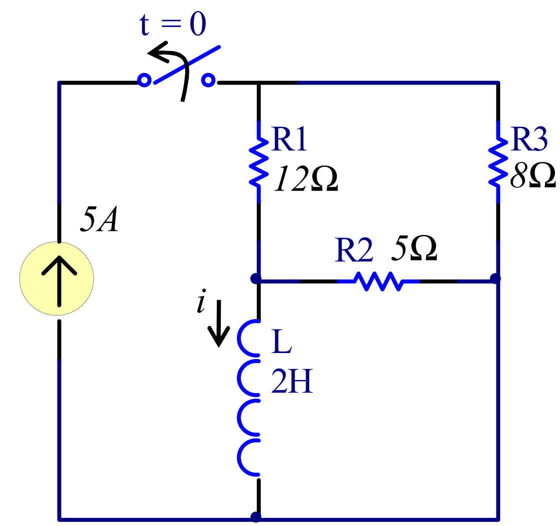
$$R_{eq} = (R_2 + R_3)//R_4 = \frac{(R_2 + R_3).R_4}{R_2 + R_3 + R_4} = 8\Omega \qquad \tau = \frac{L}{R_{eq}} = \frac{2}{8} = 0,25s$$

$$\rightarrow i(t) = i(0).e^{-t/\tau} = 6.e^{-4t}A$$

II. The source-free RC/RL circuit

II.2. R-L circuit

Ex 7.5: Calculate *i(t)* for *t > 0*.



- t < 0: the inductor acts as a short circuit

$$i(t) = \frac{R_3}{R_1 + R_3} 5 = 2A \rightarrow i(0) = i(-0) = 2A$$

- t > 0: *R-L* circuit

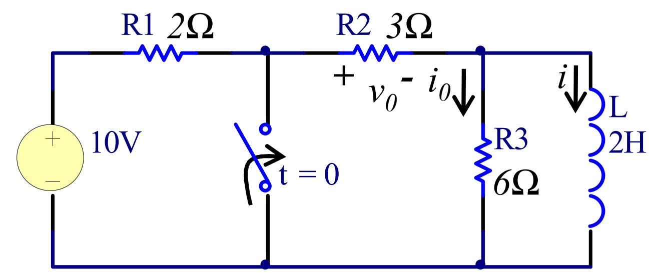
$$R_{eq} = (R_1 + R_3)//R_2 = \frac{(R_1 + R_3).R_2}{R_1 + R_2 + R_3} = 4\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{2}{4} = 0,5s$$

- Thus $i(t) = i(0).e^{-t/\tau} = 2.e^{-2t} A$

II. The source-free RC/RL circuit

II.2. R-L circuit



Ex 7.6: Find i_0, v_0 and i for all time.

- t < 0: the inductor acts as a short circuit

$$\rightarrow i_0(t) = 0 \;\; ; \;\; v_0(t) = 3.i(t) = \frac{10}{R_1 + R_2}.R_2 = 6V$$

- *t > 0:* the voltage source is short circuited

$$R_{eq} = \frac{R_2.R_3}{R_2 + R_3} = 6\Omega \;\; ; \;\; \tau = \frac{L}{R_{eq}} = 1s$$

$$\rightarrow \;\; i(t) = i(0).e^{-t/\tau} = 2e^{-t} A$$

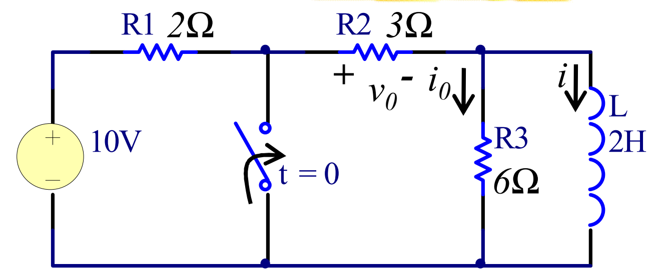
- Since the inductor is in parallel with R_2 and R_3

$$v_0(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t} V \qquad i_0(t) = \frac{v_L}{R_2} = -\frac{2}{3}e^{-t} A$$

II. The source-free RC/RL circuit

II.2. R-L circuit

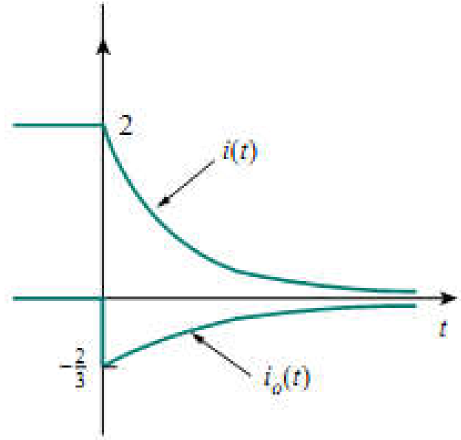
Ex 7.6: Find i_0, v_0 and i for all time.



- For all time:

$$i_0(t) = \begin{cases} 0A, & t < 0 \\ -\dfrac{2}{3}e^{-t}A, & t > 0 \end{cases} \qquad v_0(t) = \begin{cases} 6A, & t < 0 \\ 4e^{-t}V, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2A, & t < 0 \\ 2e^{-t}A, & t \ge 0 \end{cases}$$

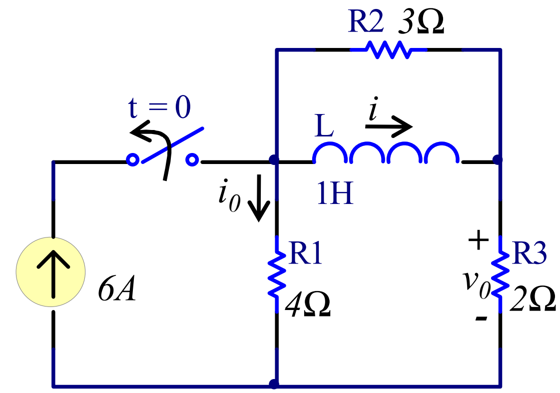


- Notes:
  - ❖ The inductor current is continuous at $t = 0$.
  - ❖ i_0(t) drops from *0* to *-2/3A, and $v_0(t)$ drops from 6 to 4* at *t = 0.*
  - ❖ τ is the same regardless of what the output is defined to be.

II. The source-free RC/RL circuit

II.2. R-L circuit

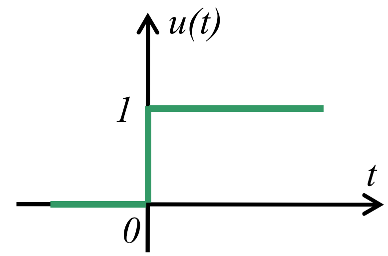
Ex 7.7: Find i_0, i, and v_0 for all time.



III. Singularity functions

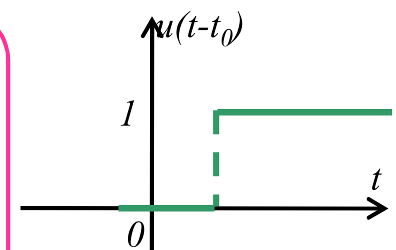
- *Singularity function* (called *switching function*) are functions that either are discontinuous or have discontinuous derivatives.

- The singularity functions are useful in circuit analysis: Serve as good approximations to the switching signals. The three most widely used singularity functions in circuit analysis: *unit step, unit impulse, unit ramp*.



- The *unit step function u(t)* is 0 for negative values of t and 1 for positive values of t.

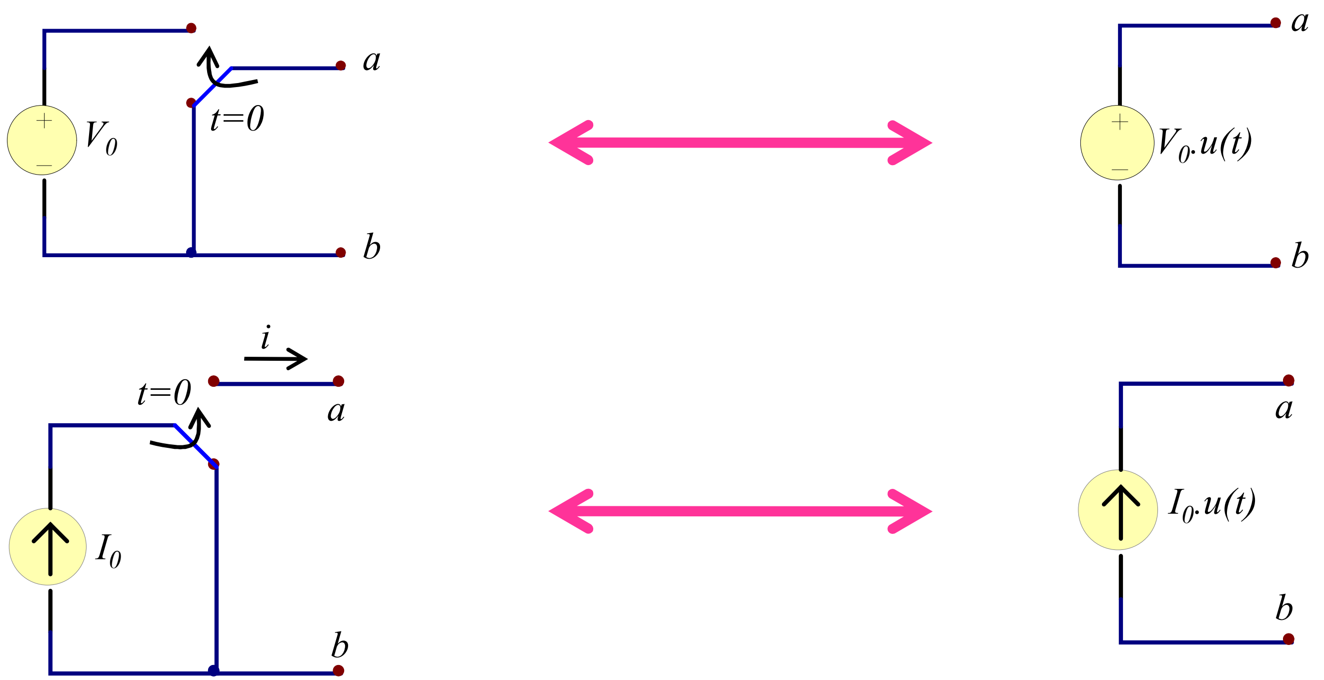
$$u(t)=\begin{cases} 0, & t<0 \\ 1, & t>0 \\ \text{undefined at } t=0 \end{cases} \qquad u(t-t_0)=\begin{cases} 0, & t<t_0 \\ 1, & t>t_0 \\ \text{undefined at } t=t_0 \end{cases}$$



III. Singularity functions

- In circuit analysis, the step function can be used to represent an abrupt change in voltage or current.

$$v(t) = \begin{cases} 0, t < t_0 \\ V_0, t > t_0 \end{cases} \leftrightarrow v(t) = V_0.u(t - t_0)$$

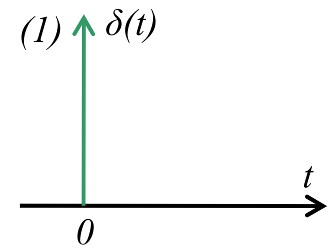


III. Singularity functions

- The derivative of the unit step function *u(t)* is the *unit impulse function δ(t)* (called *delta function*).

$$\delta(t)=\frac{d}{dt}u(t)=\begin{cases}0, & t<0\\ undefined, & t=0\\ 0, & t>0\end{cases}$$

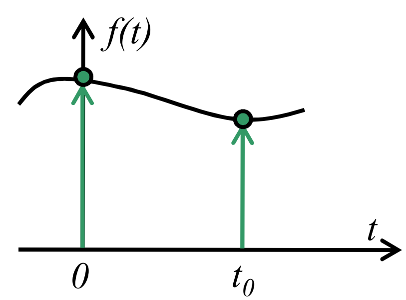
$$\int_{0-}^{0+}\delta(t)dt=1$$



- The *unit impulse function δ(t)* is zero everywhere except at *t = 0*, where it is undefined.

- Impulsive currents and voltages occur in electric circuits as a result of switching operations of impulsive sources (not physically realizable, but useful mathematical tool).

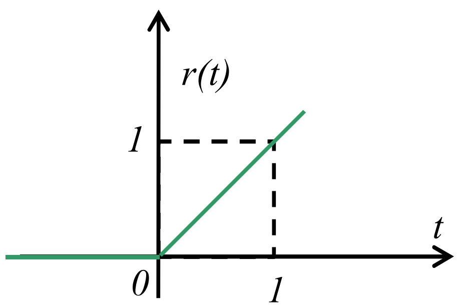
$$\int_a^b f(t)\delta(t)dt=f(0) \qquad \int_a^b f(t)\delta(t-t_0)dt=f(t_0)$$

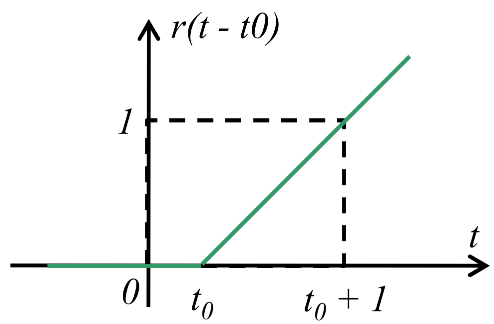


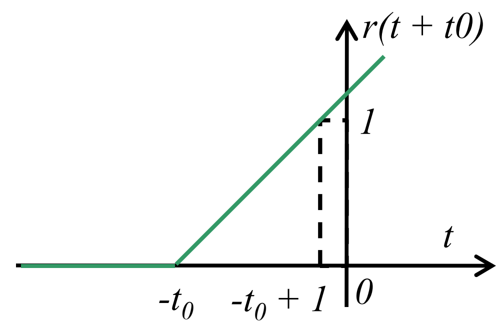
III. Singularity functions

- The *unit ramp function* is zero for negative values of *t* and has a unit slope for positive values of *t*.

$$r(t) = \int_{-\infty}^{t} u(t)dt = t.u(t)$$







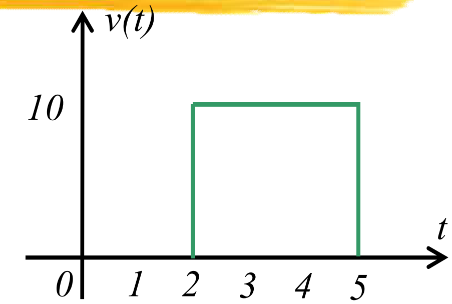
$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases}$$

$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t - t_0, & t \ge -t_0 \end{cases}$$

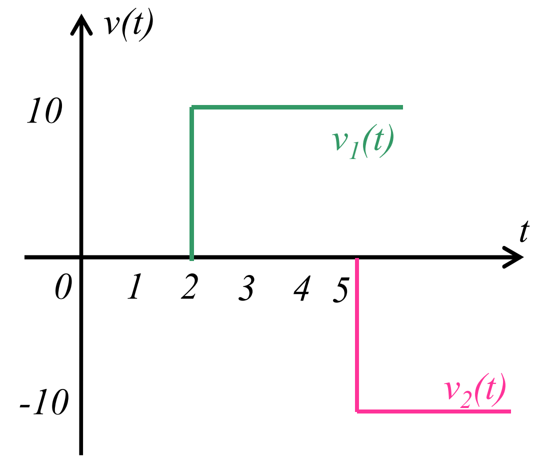
III. Singularity functions

Ex 7.8: Express the voltage pulse (*gate function*) in terms of the unit step. Calculate its derivative and sketch it.



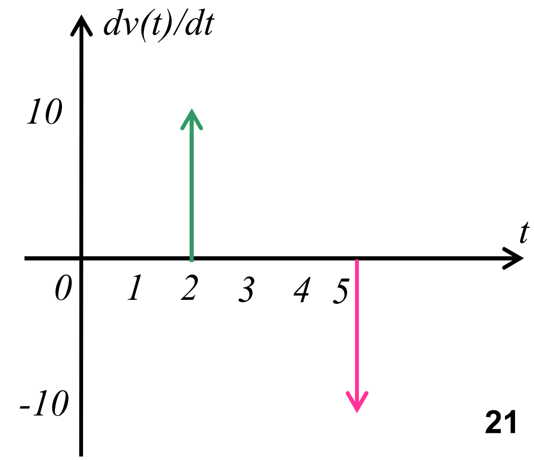
- The gate function may be regarded as a step function:
  - Switch on at one value of t $(t = 2s)$
  - Switch off at another value of t $(t = 5s)$

$$v(t) = v_1(t) + v_2(t) = 10u(t-2) - 10u(t-5)$$



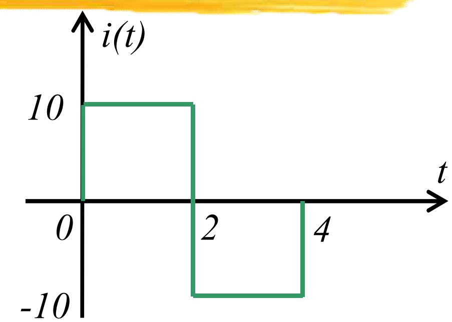
- The derivative of the gate function:

$$\frac{d}{dt}v(t) = 10\left[\delta(t-2) - \delta(t-5)\right]$$



III. Singularity functions

Ex 7.9: Express the current pulse in terms of the unit step. Find its integral and sketch it.

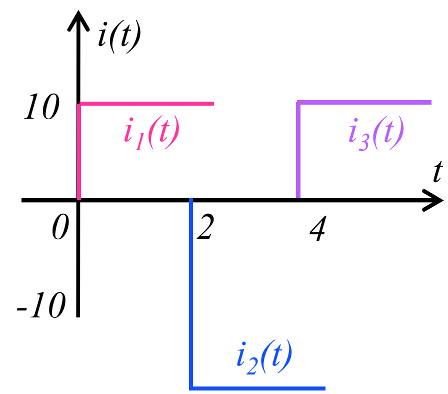


- The current pulse may be regarded as following:

$$i(t) = i_1(t) + i_2(t) + i_3(t)$$

$$i_1(t) = 10.u(t) \qquad i_2(t) = -20.u(t-2) \qquad i_3(t) = 10.u(t-4)$$

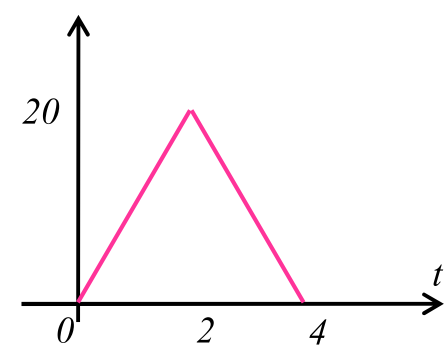
$$\rightarrow i(t) = 10.\left[u(t) - 2.u(t-2) + u(t-4)\right] A$$



- The integral of the current pulse:

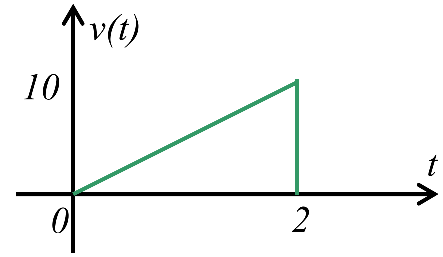
$$\int_{-\infty}^{t} i(t)dt = \int_{-\infty}^{t} 10.\left[u(t) - 2.u(t-2) + u(t-4)\right] dt$$

$$\rightarrow \int_{-\infty}^{t} i(t)dt = 10\left[r(t) - 2r(t-2) + r(t-4)\right] A$$

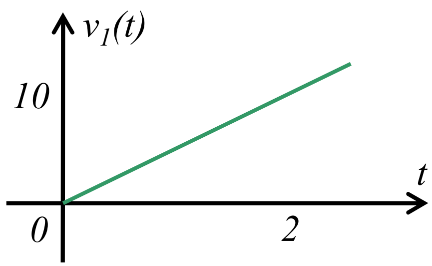


III. Singularity functions

Ex 7.10: Express the *saw tooth* in terms of the singularity function.



- *Solution 1*: The saw tooth is a combination of singularity functions:



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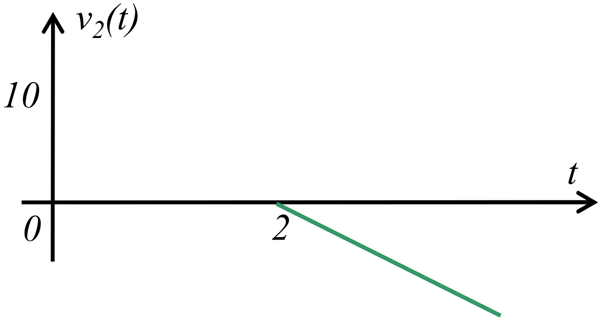

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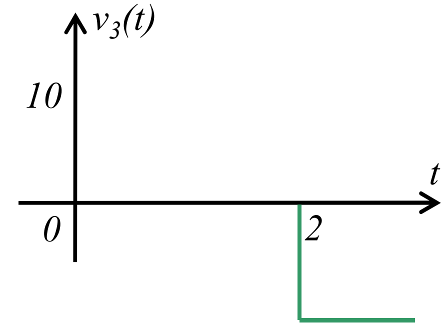


$$v_1(t) = 5.r(t) \qquad v_2(t) = -5.r(t-2) \qquad v_3(t) = -10.u(t-2)$$

- *Solution 2*: The saw tooth is a multiply of 2 functions: *u(t)* & *r(t)*

$$v(t) = 5.t.\left[u(t) - u(t-2)\right] = 5.t.u(t) - 5.t.u(t-2)$$

$$v(t) = 5.r(t) - 5.(t-2+2).u(t-2) = 5r(t) - 5r(t-2) - 10u(t-2)$$

III. Singularity functions

Ex 7.11: Express g(t) in terms of step and ramp functions. $g(t)=\begin{cases}3, & t<0\\ -2, & 0<t<1\\ 2t-4, & t>1\end{cases}$

➢ The signal *g(t)* may be regarded as the sum of three function specified within the three intervals: $g_1(t)$, $g_2(t)$, $g_3(t)$

❖ With *t < 0:* $g_1(t)=3.u(-t)=\begin{cases}3, t<0\\ 0, t>0\end{cases}$

❖ With *t > 1:* $g_3(t)=\left(2t-4\right)u(t-1)$

❖ With 0 < *t < 1:* $g_2(t)=-2\left[u(t)-u(t-1)\right]$

❖ Thus:

$$g(t)=3u(-t)-2\left[u(t)-u(t-1)\right]+(2t-4)u(t-1)$$

$$g(t)=3u(-t)-2u(t)+2(t-1)u(t-1)$$

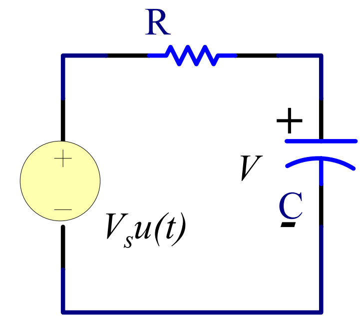
$$g(t)=3u(-t)-2u(t)+2r(t-1)\xrightarrow{u(-t)=1-u(t)}3-5u(t)+2r(t-1)$$

IV. Step response of an RC/RL circuit

IV.1. R-C circuit

- When the *DC* source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response.*

- *Step response* of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



Ex: Consider the *RC* circuit, assume an initial voltage V0 on the capacitor.

$$V(-0) = V(+0) = V_0$$

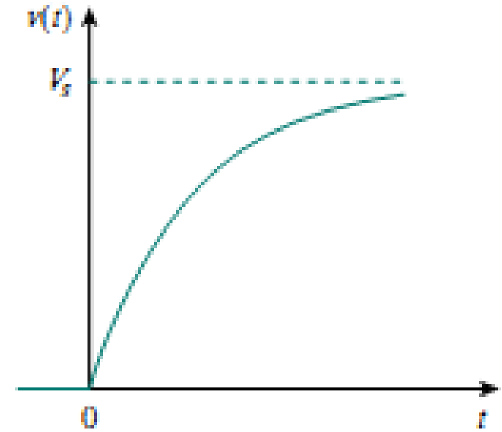
V(-0): Voltage across *C* just before switching

V(+0): Voltage across *C* immediately after switching

- Applying KCL: $C\dfrac{dv}{dt} + \dfrac{v - V_S u(t)}{R} = 0 \xrightarrow{t>0} \dfrac{dv}{dt} + \dfrac{v}{RC} = \dfrac{V_S}{RC}$

IV. Step response of an RC/RL circuit

IV.1. R-C circuit



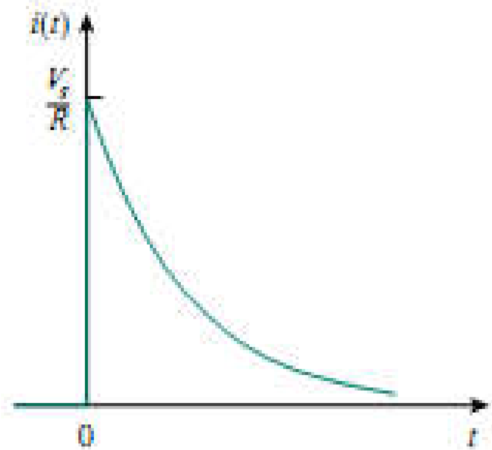
$$\frac{dv}{dt} = \frac{v - V_S}{RC} \leftrightarrow \frac{dv}{v - V_S} = -\frac{dt}{RC} \quad \rightarrow \ln(v - V_S)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\rightarrow \ln\left[v(t) - V_S\right] - \ln(V_0 - V_S) = -\frac{t}{RC} + 0$$

$$\rightarrow \ln\frac{v - V_S}{V_0 - V_S} = -\frac{t}{RC} \rightarrow \frac{v - V_S}{V_0 - V_S} = e^{-t/\tau} \rightarrow v(t) = \begin{cases} V_0, & t < 0 \\ V_S + (V_0 - V_S)e^{-t/\tau}, & t > 0 \end{cases}$$

- If C is uncharged initially $V_0 = 0$

$$\boxed{\begin{aligned} v(t) &= V_S(1 - e^{-t/\tau})u(t) \\ i(t) &= C\frac{dv}{dt} = \frac{V_S}{R}e^{-t/\tau}.u(t) \end{aligned}}$$



IV. Step response of an RC/RL circuit

IV.1. R-C circuit

- In general, for $t > 0$: $v(t) = V_S + (V_0 - V_S)e^{-t/\tau} = v_f + v_n$ *with* $\begin{cases} v_n = V_S \\ v_f = (V_0 - V_S)e^{-t/\tau} \end{cases}$

  - v_n: Natural response (*transient response*) of circuit, will decay to zero after five time constants.
  - v_f: Forced response (*steady-state response*) of circuit, will represents what the circuit is forced to do by the input excitation, and remains a long time.

- Thus, to find the step response of an RC circuit requires 3 things:

$$\boxed{v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau}}$$

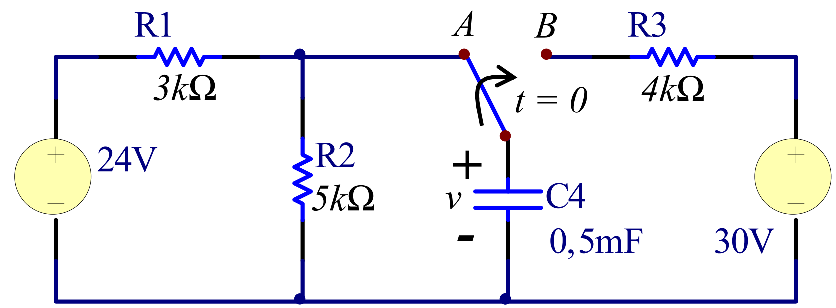
Where: *v(0)* is the initial voltage at $t = +0$.

v(∞) is the final or steady state value.

τ is the time constant of the circuit.

IV. Step response of an RC/RL circuit

IV.1. R-C circuit



Ex 7.12: The switch has been in position *A* for a long time. At *t = 0*, the switch moves to *B*.

Determine $v(t)$ for *t > 0* and calculate its value at *t = 1s* and *4s*.

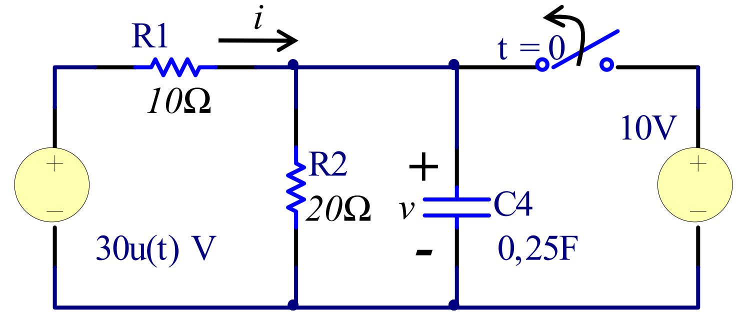
- The voltage across the capacitor at t = -0: $v(-0) = \dfrac{24}{R_1 + R_2} R_2 = 15V$
- The capacitor voltage cannot change instantaneously: $\rightarrow v(+0) = v(-0) = 15V$
- For t > 0: The time constant is: $\tau = R_3.C = 4.10^3.0,5.10^{-3} = 2s$
- Since the capacitor acts like an open circuit to *DC* at steady state: $v(\infty) = 30V$

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau} = 30 + (15-30)e^{-t/\tau} = 30 - 15e^{-0,5t}\,(V)$$

- At *t = 1:* $v(1s) = 30 - 15e^{-0,5} = 20,902(V)$
- At *t = 4:* $v(4s) = 30 - 15e^{-2} = 27,97(V)$

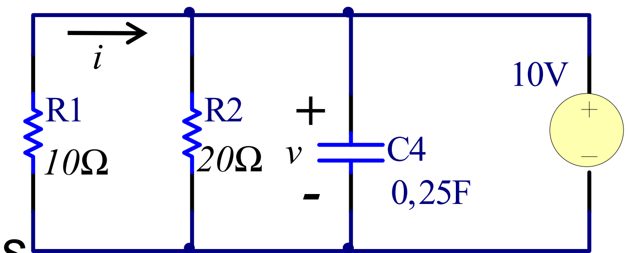
IV. Step response of an RC/RL circuit

IV.1. R-C circuit



Ex 7.13: The switch has been closed for a long time and is opened at $t = 0$. Find i, v for all time.

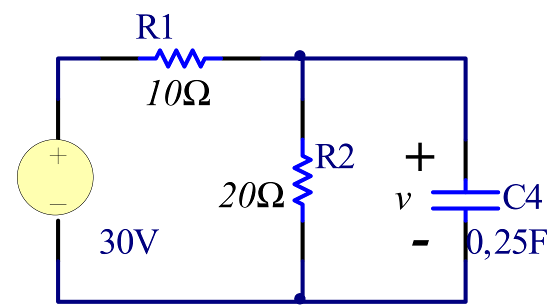
- At $t = 0$: i can be discontinuous but v cannot → Find v and then obtain i from v.



- For t < 0: $v(0) = v(-0) = 10V, \quad i = -\dfrac{v}{R_1} = -1A$
- For t > 0:

$$v(\infty) = \frac{R_2}{R_1 + R_2} 30 = 20V \quad \tau = R_{eq}.C = \frac{R_1.R_2}{R_1 + R_2}.C = \frac{5}{3}s$$

Thus: $v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau} = (20 - 10e^{-0,6t})V$



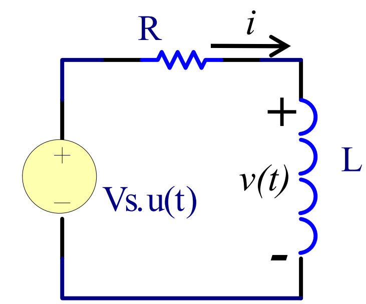
The current i is the sum of the currents through R_2 and C

$$i(t) = \frac{v}{R_2} + C\frac{dv}{dt} = 1 - 0,5e^{-0,6t} + 0,25(-0,6)(-10)e^{-0,6t}$$

$$i(t) = 1 + e^{-0,6t} A$$

IV. Step response of an RC/RL circuit

IV.2. R-L circuit

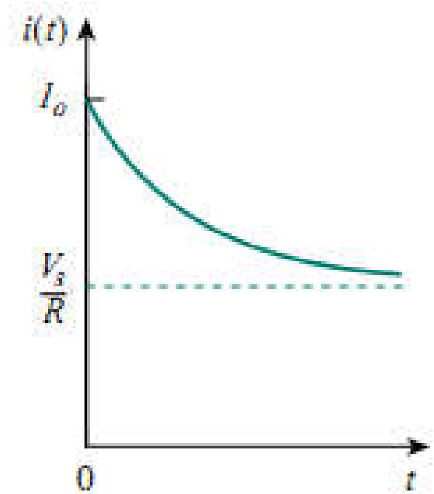


- Consider the *R-L* circuit → find i as the circuit response
- Let i be the sum of the natural current and the forced current

$$i = i_n + i_f$$

- ❖ The natural response: $i_n = A.e^{-t/\tau}, \quad \tau = \dfrac{L}{R}$
- ❖ The forced response: $i_f = \dfrac{V_S}{R}$

$$\left.\begin{matrix} i_n = A.e^{-t/\tau}, \ \tau = \dfrac{L}{R} \\ i_f = \dfrac{V_S}{R} \end{matrix}\right\} \rightarrow i(t) = A.e^{-t/\tau} + \frac{V_S}{R}$$

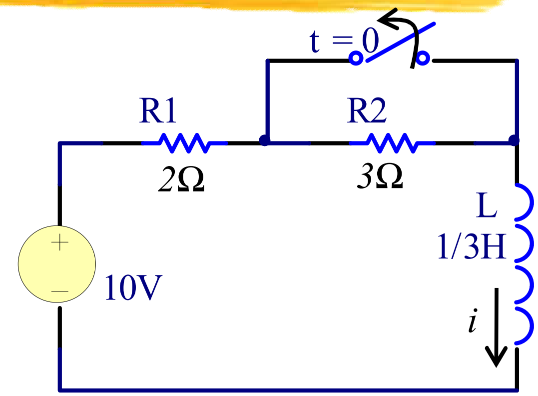


- Let I_0 be the initial current through *L:* $i(+0) = i(-0) = I_0$
- Thus at *t = 0:* $I_0 = A + \dfrac{V_S}{R} \rightarrow A = I_0 - \dfrac{V_S}{R}$
- The complete response of the RL circuit:

$$\boxed{i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R}\right)e^{-t/\tau} = i(\infty) + \left[i(0) - i(\infty)\right]e^{-t/\tau}}$$

IV. Step response of an RC/RL circuit

IV.2. R-L circuit



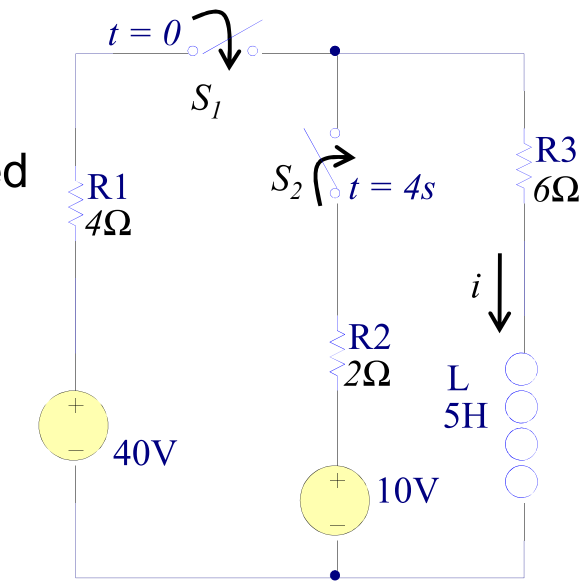
Ex 7.14: Find i(t) for t > 0. Assume that the switch has been closed for a long time.

- For *t < 0:* $i(-0) = \dfrac{10}{R_1} = 5A$
- Since the i_L cannot change instantaneously: $i(0) = i(+0) = i(-0) = 5A$
- For *t > 0:* $i(\infty) = \dfrac{10}{R_1 + R_2} = 2A$
- The time constant: $\tau = \dfrac{L}{R_{eq}} = \dfrac{L}{R_1 + R_2} = \dfrac{1}{15}s$
- Thus: $i(t) = i(\infty) + \left[i(0) - i(\infty)\right]e^{-t/\tau}$

$$i(t) = 2 + (5-2)e^{-15t} = 2 + 3e^{-15t}A, \quad t > 0$$

IV. Step response of an RC/RL circuit

IV.2. R-L circuit



Ex 7.15: At $t = 0$, switch 1 is closed, and switch 2 is closed 4s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$

- We need to consider the 3 time intervals:
  - For $t < 0$: Two switches are open

$$i(-0) = i(0) = i(+0) = 0$$

  - For $0 \le t \le 4$: The switch S_1 is closed, S2 is open.

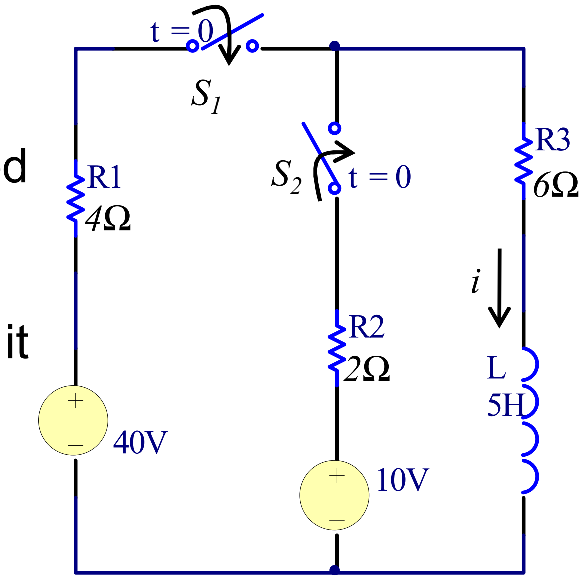
$$i(\infty) = \frac{40}{R_1 + R_3} = 4A, \quad R_{eq} = R_1 + R_3 = 10\Omega, \quad \tau = \frac{L}{R_{eq}} = 0,5s$$

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right]e^{-t/\tau} = 4 + (0-4)e^{-2t} = 4(1 - e^{-2t})A$$

IV. Step response of an RC/RL circuit

IV.2. R-L circuit

Ex 7.15: At $t = 0$, switch 1 is closed, and switch 2 is closed $4s$ later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$



- For $t \geq 4$: S_2 is closed, but not affect i_L because it cannot change abruptly. Thus the initial current is

$$i(4) = i(-4) = 4(1 - e^{-8}) \simeq 4A$$

To find i(∞), using KCL:

$$\frac{40 - v}{R_1} + \frac{10 - v}{R_2} = \frac{v}{R_3} \rightarrow v = \frac{180}{11} V \rightarrow i(\infty) = \frac{v}{6} = 2,727A$$

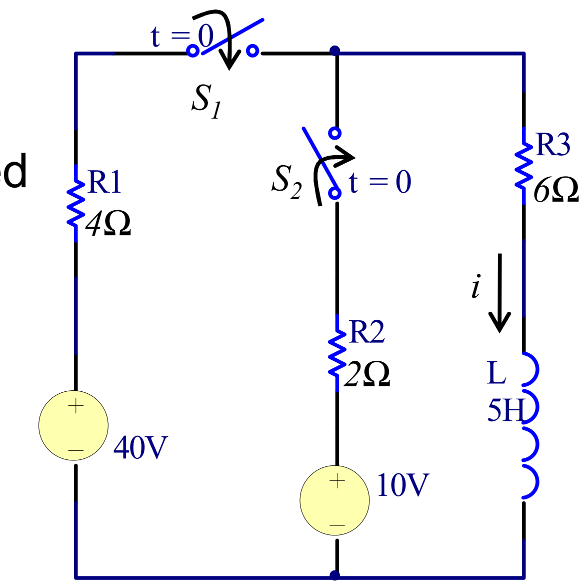
$$R_{eq} = (R_1 // R_2) + R_3 = \frac{22}{3}\Omega \quad \rightarrow \tau = \frac{L}{R_{eq}} = \frac{15}{22}s$$

$$i(t) = i(\infty) + \left[i(4) - i(\infty)\right]e^{-(t-4)/\tau} = 2,727 + 1,273e^{-1,4667(t-4)}, t \geq 4$$

IV. Step response of an RC/RL circuit

IV.2. R-L circuit

Ex 7.15: At *t = 0*, switch 1 is closed, and switch 2 is closed *4s* later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$



- Putting all this together:

$$i(t)=\begin{cases}0, & t\le 0\\ 4(1-e^{-2t}), & 0\le t\le 4\\ 2,727+1,273e^{-1,4667(t-4)}, & t\ge 4\end{cases}$$

- At $t = 2s$: $i(2)=4(1-e^{-4})=3,93A$

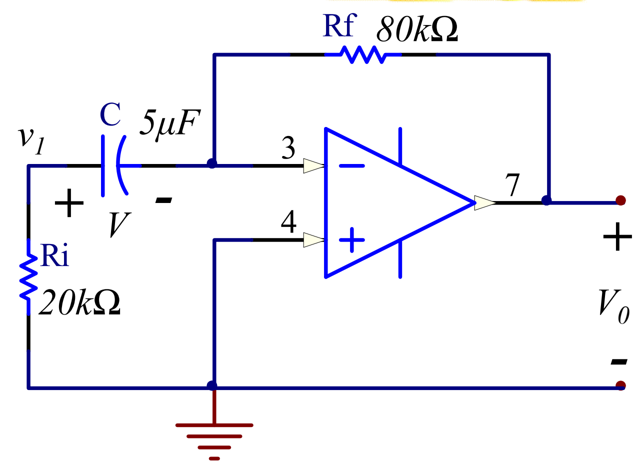
- At $t = 5s$: $i(5)=2,727+1,273e^{-1,4667}=3,02A$

V. First-order Op Amp circuits

- An op amp circuit containing a storage element will exhibit first-order behaviors
- Differentiators and integrators treated in Chapter 5 are examples of first-order op amp circuits.
- For practical reasons, inductors are hardly ever used in op amp circuits, therefore, the op amp circuits we consider here are of the RC type.
- Methods to analyze op amp circuits:
  - Using nodal analysis.
  - Using the Thevenin equivalent circuit to simpler the op amp circuit

V. First-order Op Amp circuits

Ex 7.16: Find v_0 for $t > 0$, give that $v(0) = 3V$



- *Solution 1:*
  - Applying KCL gives: $\dfrac{-V_1}{R_1} = C\dfrac{dV}{dt}$
  - Node 3 and 4 must be at the same potential

$$V_1 - 0 = V \rightarrow V_1 = V \rightarrow \frac{dV}{dt} + \frac{V}{CR_1} = 0$$

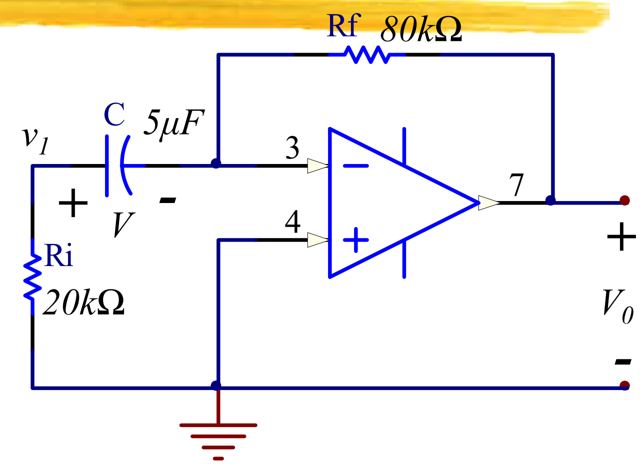
$$\rightarrow V(t) = V_0 e^{-t/\tau}, \quad \tau = R_1 C \quad \rightarrow V(t) = 3e^{-10t}$$

  - At $t > 0$, applying KCL at node 3 gives:

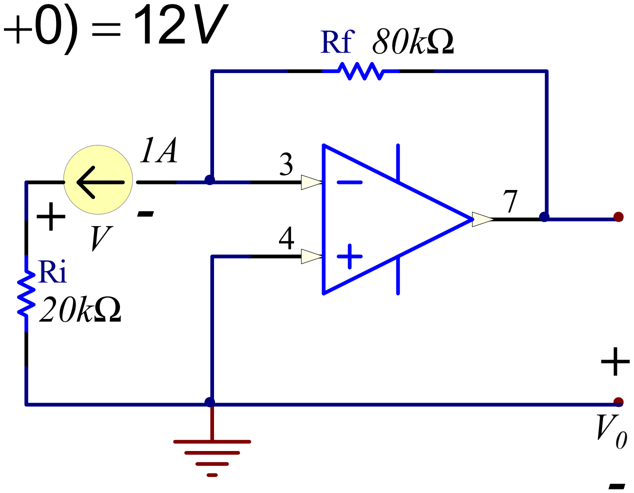
$$C\frac{dV}{dt} = \frac{0 - V_0}{R_f} \rightarrow V_0 = -R_f C\frac{dV}{dt} = -80.10^3.5.10^{-6}(-30e^{-10t}) = 12e^{-10t}\,V$$

V. First-order Op Amp circuits

Ex 7.16: Find v_0 for $t > 0$, give that $v(0) = 3V$



- *Solution 2:* Apply the short-cut method
  - Voltage across C: $V(+0) = V(-0) = 3V$
  - Apply KCL at node 3: $\dfrac{3}{R_i} + \dfrac{0 - V_0(+0)}{R_f} = 0 \rightarrow V_0(+0) = 12V$



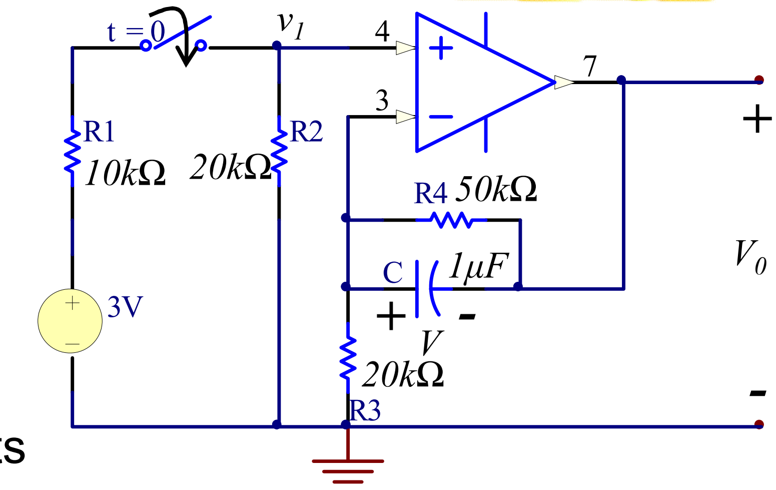
  - The circuit is source free → $v(\infty) = 0V$
  - To find τ, need calculate R_{eq} across C
    - Remove, replace C by a 1-A current source.
    - Applying KVL to the input loop yields

$$20.10^3.1 - v = 0 \rightarrow v = 20kV \rightarrow R_{eq} = \frac{v}{1} = 20k\Omega \quad \rightarrow \tau = R_{eq}C = 0{,}1s$$

$$V_0(t) = V_0(\infty) + \left[V_0(0) - V_0(\infty)\right]e^{-t/\tau} = 12e^{-10t}V, \quad t > 0$$

V. First-order Op Amp circuits

Ex 7.17: Determine *v(t) and* $v_0(t)$



- *V(t)* is the step response

$$V(t) = V(\infty) + \left[V(0) - V(\infty)\right]e^{-t/\tau}, \quad t > 0$$

- No current enters the op amp → the elements on the feedback loop constitute an *RC* circuit $\rightarrow \tau = RC = 50.10^3.10^{-6} = 0,05s$
- For *t < 0*: Switch is open → *V(0) = 0.*
- For *t > 0*: $\rightarrow v_1 = \dfrac{R_2}{R_1 + R_2}3 = 2V$
- At steady state: C acts like an open circuit → op amp is a non-inverting amplifier

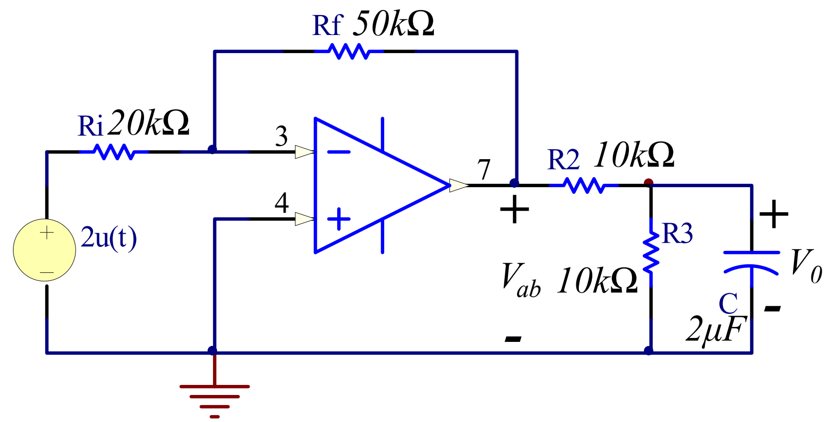
$$V_0(\infty) = (1 + \frac{R_4}{R_3})V_1 = 3,5.2 = 7V \xrightarrow{V_1 - V_0 = V} V(\infty) = 2 - 7 = -5V$$

- Thus: $V(t) = -5 + \left[0 - (-5)\right]e^{-20t} = 5(e^{-20t} - 1)V, \ \ t > 0$

$$V_0(t) = V_1(t) - V(t) = 7 - 5e^{-20t}V, \ \ t > 0$$

V. First-order Op Amp circuits

Ex 7.18: Find the step response $v_0(t)$ for $t > 0$.



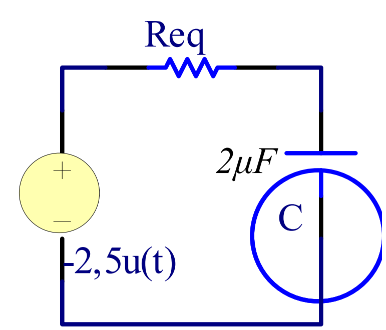
- Remove C, and find the Thevenin equivalent at its terminal.
  - Open voltage at the terminal:

$$V_{ab} = -\frac{R_f}{R_I}V_I \rightarrow V_{Th} = \frac{R_3}{R_2+R_3}V_{ab} = -\frac{R_3}{R_2+R_3}\frac{R_f}{R_I}V_I = -2,5u(t)$$

  - Thevenin resistance: $R_{Th} = (R_0 + R_2)//R_3 \xrightarrow[ideal\ op\ amp]{R_0 \simeq 0} \frac{R_2R_3}{R_2+R_3} = 5k\Omega$

- Gives the Thevenin equivalent circuit:

$$V_0(t) = -2,5(1-e^{-t/\tau})u(t), \quad \tau = R_{Th}.C = 5.10^3.2.10^{-6} = 0,01s$$

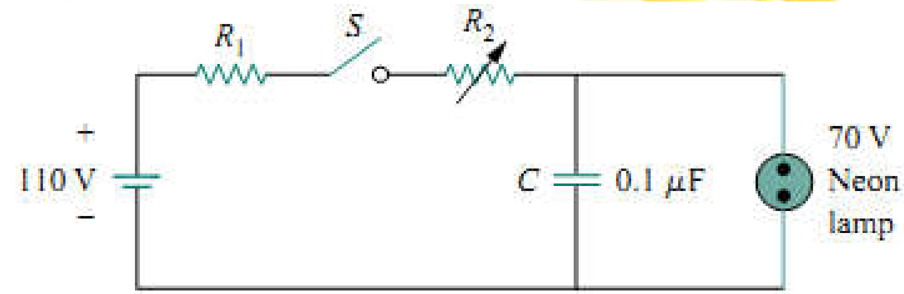


- Thus, the step response for t > 0 is:

$$V_0(t) = 2,5(e^{-100t}-1)u(t)\,V$$

VI. Applications

VI.1. Delay circuits



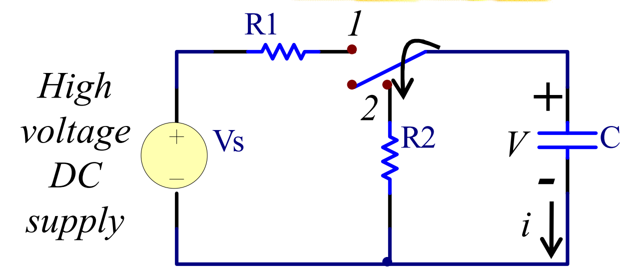
Ex7.19: An RC circuit with capacitor connected in parallel with a neon lamp.

The voltage source can provide enough voltage to fire the lamp.

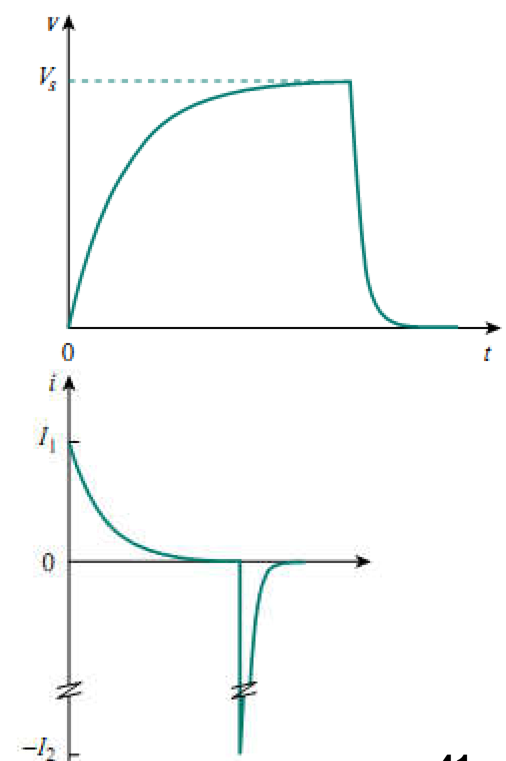
- Switch closes: V_C increases to *110V* with the time constant $(R_1 + R_2)C$.
- The lamp will act as an open-circuit and not emit light until the voltage across it exceeds a particular level (*70V*).
- When V_C reaches, the lamp fires and the capacitor discharges through it → V_C drops and the lamp turn off.
- The lamp acts again as an open-circuit and *C* recharges: Adjusting R_2, we can introduce either short or long time delays.

VI. Applications

VI.2. Photoflash unit



- This application exploits the ability of the capacitor to oppose any abrupt change in voltage.
- Switch is in 1: *C* charges slowly due to the large time constant $\tau = R_1C$
  - V_C rises gradually from zero to V_S
  - I_C *decreases from* I_1 to *0*



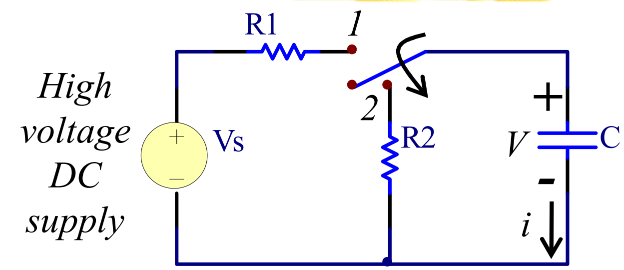
- Switch is in 2: *C dis*charges
  - Low resistance R_2 of the photo-lamp permits a high discharge current with peak I_2 in a short duration.

$$t_{discharge} = 5R_2C$$

- The simple *RC* circuit provides a short-duration, high current pulse.

VI. Applications

VI.2. Photoflash unit



Ex 7.20: An electronic flashgun has a current limiting $R_1 = 6k\Omega$, and C = $2000\mu F$ charged to 240V. If the lamp resistance R_2 is 12Ω. Find:

a. Peak charging current: $I_1 = \frac{V_S}{R_1} = \frac{240}{6.10^3} = 40mA$

b. Time required for the C to fully charge: $t_{charge} = 5R_1C = 5.6.10^3.2000.10^{-6} = 60s$

c. Peak discharging current: $I_1 = \frac{V_S}{R_2} = \frac{240}{12} = 20A$

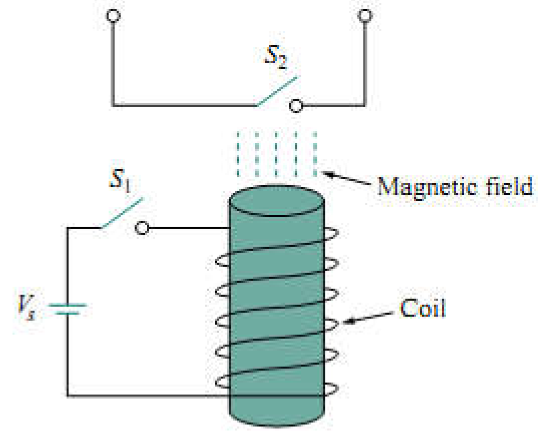
d. Energy stored: $W = \frac{1}{2}CV_S^2 = \frac{1}{2}.2000.10^{-6}.240^2 = 57,6J$

e. Energy stored in C is dissipated across the lamp during the discharging period:

$$t_{discharge} = 5R_2C = 5.12.2000.10^{-6} = 0,12s \quad \rightarrow p = \frac{W}{t_{discharge}} = \frac{57,6}{0,12} = 480W$$

VI. Applications

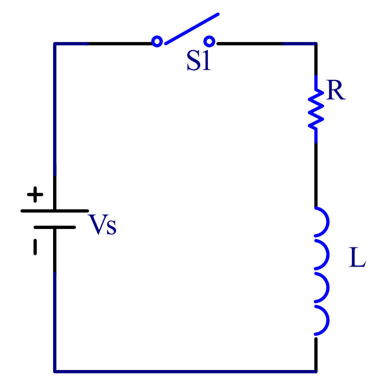
VI.3. Relay circuits



- A *relay* is essentially an electromagnetic device used to open or close a switch that controls another circuit.

- The coil circuit is an *RL* circuit:

  - When S_1 is closed → i_L increases, produces a magnetic field
  - The magnetic field is sufficiently strong to pull the movable contact in the other circuit and close switch S_2.
  - Time interval t_d between the closure of switches S_1 and S_2 is called the *relay delay time*.



- Relays were used in the earliest *digital circuits and* are still used for *switching high power circuits*.

VI. Applications

VI.3. Relay circuits

Ex 7.21: The coil of a certain relay is operated by a 12V battery. If the coil has a resistance of 150Ω and an inductance of 30mH and the current needed to pull in is 50mA. Calculate the relay delay time.

- The current through the coil is given by: $i(t) = i(\infty) + \left[i(0) - i(\infty)\right]e^{-t/\tau}$

where: $i(0) = 0,\quad i(\infty) = \dfrac{V_S}{R_L} = \dfrac{12}{150} = 80mA,\quad \tau = \dfrac{L}{R} = \dfrac{30.10^{-3}}{150} = 0,2ms$

- Thus: $i(t) = 80\left[1 - e^{-t/\tau}\right]mA$

- If $i(t_d)$ = *50mA*, then: $50 = 80\left[1 - e^{-t/\tau}\right] \rightarrow e^{t_d/\tau} = \dfrac{8}{3}$

$$\rightarrow t_d = \tau \ln\frac{8}{3} = 0,2.\ln\frac{8}{3} = 0,1962ms$$