Chapter 4: Circuit theorems

I. Introduction.

II. Linearity property.

III. Superposition.

VI. Source transformation.

V. Thevenin’s theorem.

VI. Norton’s theorem.

VII. Maximum power transfer

Chapter 4: Circuit theorems

I. Introduction

- Chapter 3 presented some analysis method using Kirchhoff's laws
  - Advantage: Analyze a circuit without changing the its configuration.
  - Disadvantage: For a large, complex circuit → it's hard to compute, and solve the set of equations.
- For complex circuits, it need to develop some theorems to simplify circuit analysis, such as Thevenin's and Norton's theorems (*applicable only to linear circuits*).
- This chapter presents:
  - Concept of circuit linearity.
  - Circuit theorems.
  - Concept of superposition, source transformation, maximum power transfer.

II. Linearity property

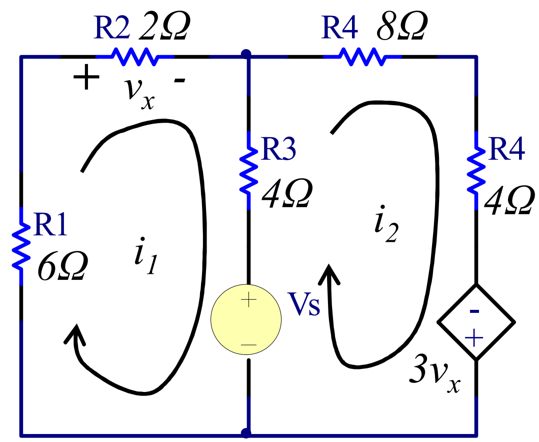
- ***Linearity*** *is the property of an element describing a linear relationship between cause (input, excitation) and effect (output, response)*
- Linearity property combines:
  - *Homogeneity (scaling) property*: $v = iR \rightarrow kiR = kv$
  - *Additivity property:* $\begin{matrix} v_1 = i_1 R \\ v_2 = i_2 R \end{matrix} \xrightarrow{i_1 + i_2} v = (i_1 + i_2)R = i_1 R + i_2 R = v_1 + v_2$

Ex: Resistor is a linear element because the voltage-current relationship satisfies both the homogeneity and additivity properties.

> A ***linear circuit*** is one whose output is linearly related (or directly proportional) to its input.

- A linear circuit consists of only linear elements, linear dependent sources, and independent sources.

II. Linearity property



Ex 4.1: Find i_0 when v_s = *12V* and v_s = *24V*.

- Applying KVL to the 02 loops gives:

$$\begin{cases} 12i_1 - 4i_2 + v_S = 0 \\ -4i_1 + 16i_2 - 3v_x - v_S = 0 \end{cases} \xrightarrow{v_x = 2i_1} \begin{cases} 12i_1 - 4i_2 + v_S = 0 \\ -10i_1 + 16i_2 - v_S = 0 \end{cases}$$

- Solving the set of equations gives:

$$\begin{cases} i_1 = -6i_2 \\ i_2 = \dfrac{v_S}{76} \end{cases} \xrightarrow{v_S = 12V} i_0 = i_2 = \frac{12}{76} = 0.158A$$

- Because, this circuit is linear circuit → applying the linearity property gives:

$$v_S = 24V \rightarrow i_0 = i_2 = 2.0,158 = 0,316A$$

III. Superposition

- If circuit has two or more independent sources, there several ways to determine the value of a specific variable (voltage, current):
    - Use nodal or mesh analysis.
    - *Superposition approach*: Determine the contribution of each independent source to the variable, and then add them up.

The ***superposition*** principle states that the voltage across (or current through) an element ***in a linear circuit*** is the algebraic sum of the voltages across (or currents though) that element due to each independent source acting alone.

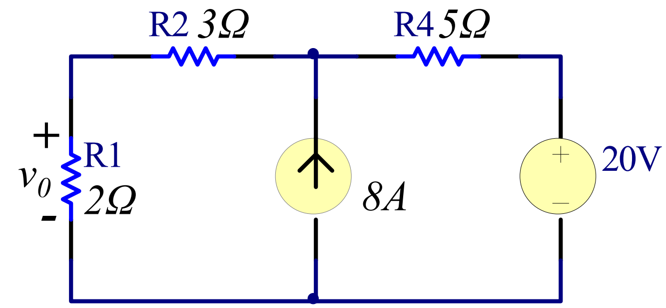
- Superposition is not limited to circuit analysis but is applicable in many field where cause and effect bear a linear relationship to one another.

III. Superposition

- Step to apply superposition principle:
  - Turn off all ***independent sources*** except one source (*dependent sources are left intact*):
    - Replace ***voltage source*** by ***short circuit***
    - Replace ***current source*** by ***open circuit***
  - Find the output (voltage or/and current) due to that active source (using nodal or mesh analysis)
  - Repeat step 1 & 2 for each of the other independent sources.
  - Find the total contribution by adding algebraically all the contributions due to the independent sources.

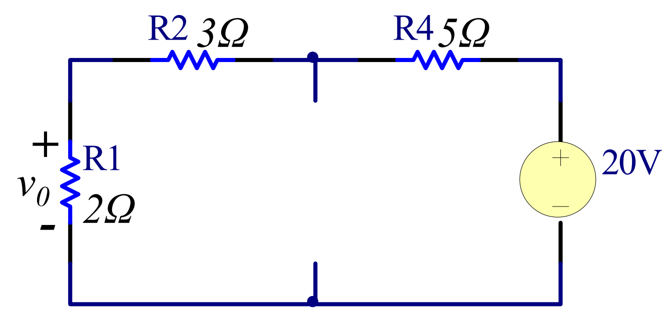
III. Superposition

Ex 4.2: Using the superposition theorem, find v_0.



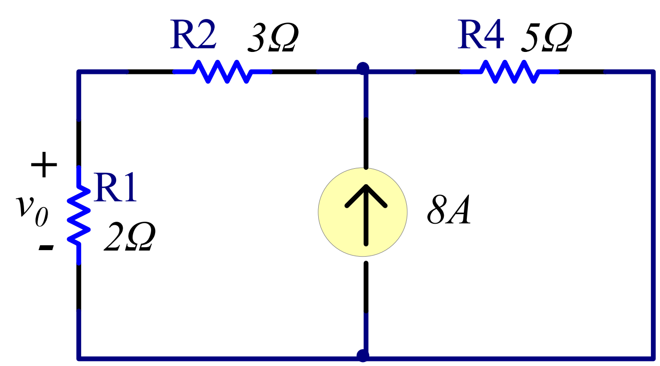
- Since there are two sources, let: $v_0 = v_{01} + v_{02}$
- To obtain v_{01}, set the current source to zero
  - Applying KVL to the loop gives:

$$(3+5+2)i = 20 \rightarrow i = 2A \rightarrow v_{01} = 2.2 = 4V$$



- To obtain v_{02}, set the voltage source to zero
  - Using current division gives

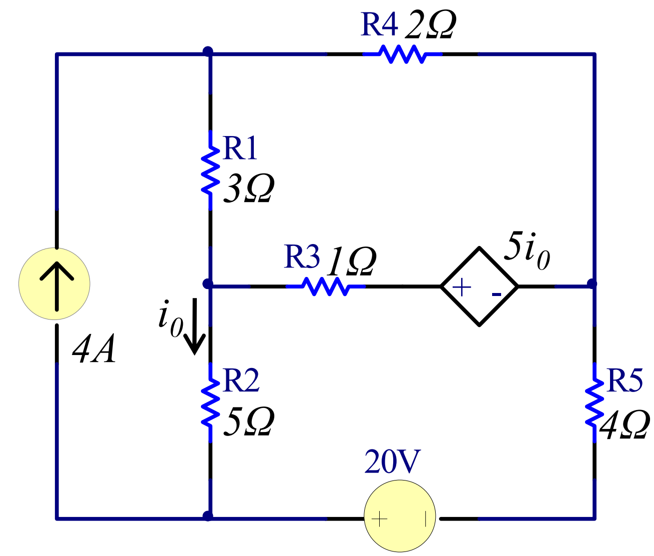
$$i_{R_1} = \frac{8}{2+3+5}5 = 4A \;\rightarrow v_{02} = 2.4 = 8V$$



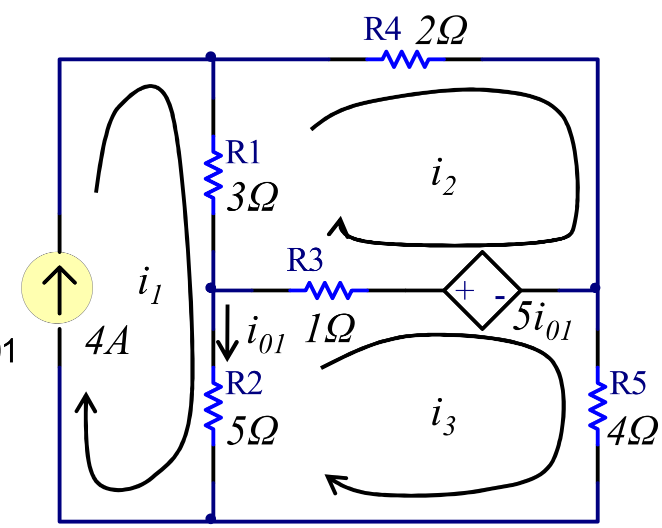
- Finally, we find: $v_0 = v_{01} + v_{02} = 8 + 4 = 12V$

III. Superposition

Ex 4.3: Using the superposition theorem, find i_0.



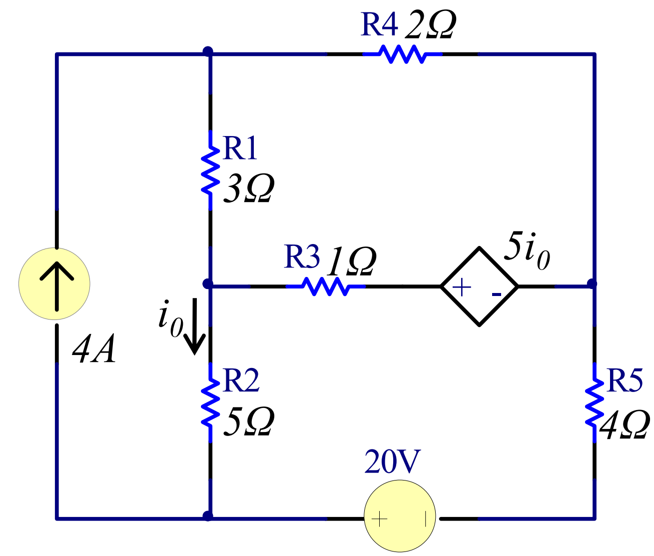
- In this circuit, there is a dependent source, we left intact. We let: $i_0 = i_{01} + i_{02}$
- To obtain i_{01}, turn off the 20-V source to zero
  - Applying mesh analysis, we have:
    - Loop 1: $i_1 = 4A$
    - Loop 2: $-3i_1 + 6i_2 - i_3 - 5i_{01} = 0$
    - Loop 3: $-5i_1 - i_2 + 10i_3 + 5i_{01} = 0$
    - Applying KCL at node 0: $i_3 = i_1 - i_{01} = 4 - i_{01}$



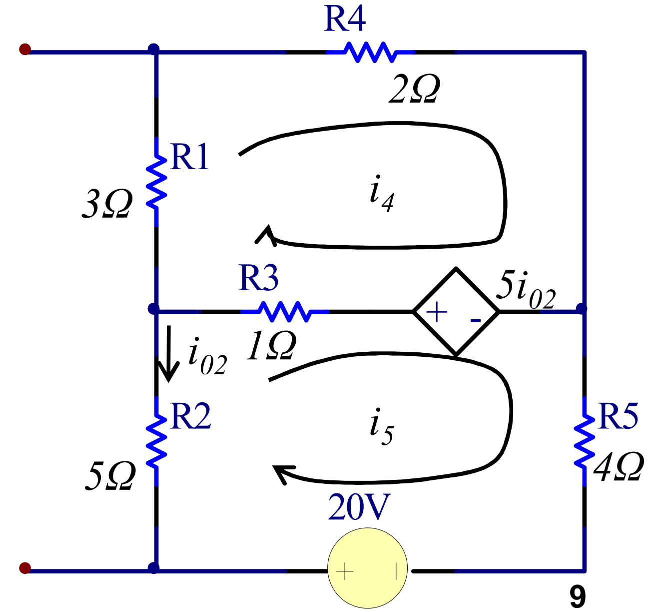
- From the four equations, we have: $i_{01} = 3,06A$

III. Superposition

Ex 4.3: Using the superposition theorem, find i_0.

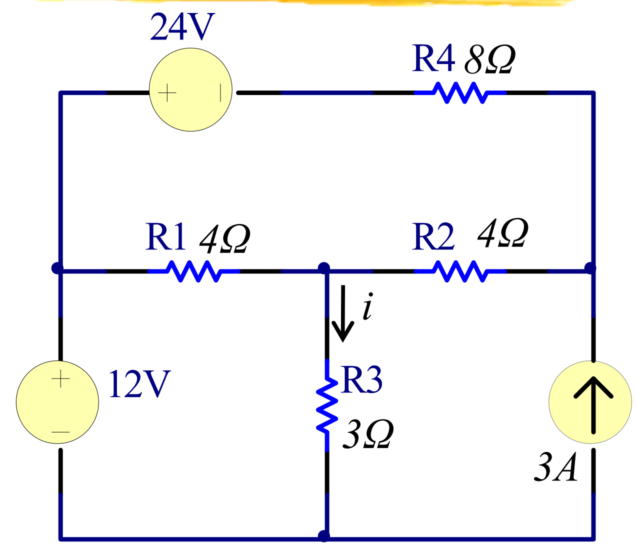


- To obtain i_{02}, turn off the 4-A source to zero
  - ❖ Applying KVL, we have:
    - ❑ Loop 4: $6i_4 - i_5 - 5i_{02} = 0$
    - ❑ Loop 5: $-i_4 + 10i_5 - 20 + 5i_{02} = 0$
    - ❑ In loop 5, we have: $i_5 = -i_{02}$
- From the four equations, we have: $i_{02} = -3,53A$
- Finally, when there are the both sources in the circuit, we find: $i_0 = i_{01} + i_{02} = -0,47A$



III. Superposition

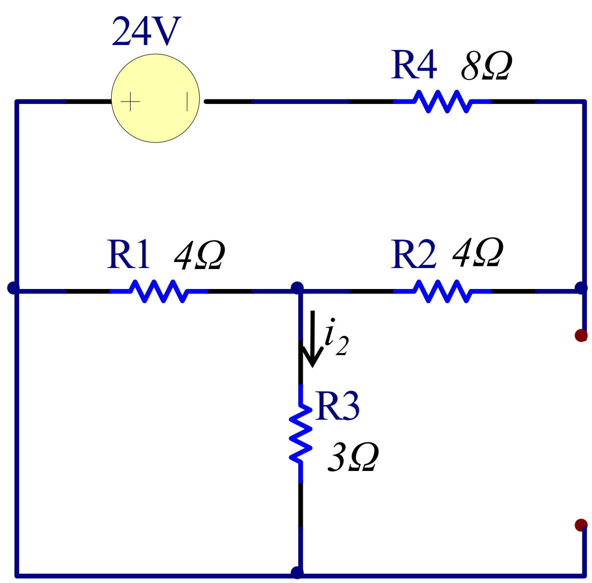
Ex 4.4: Using the superposition theorem, find *i*.

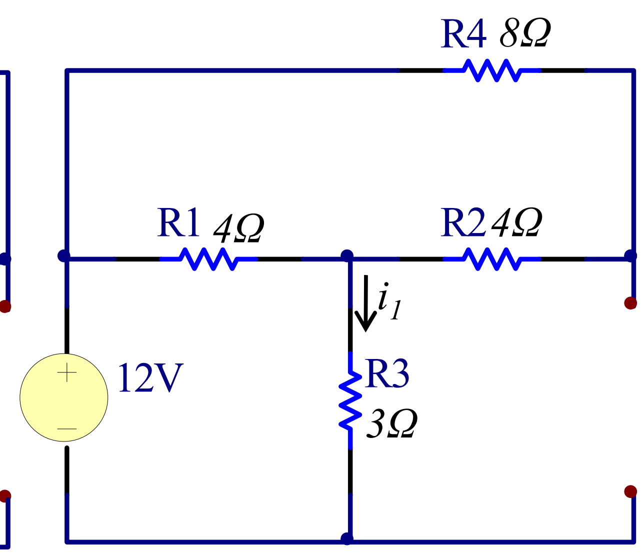


- The circuit has three sources, let: $i = i_1 + i_2 + i_3$
- Getting i_1, turn off 3-A source, 24-V source

$$i_1 = \frac{12}{\left[R_1 // (R_2 + R_4)\right] + R_3} = \frac{12}{6} = 2A$$

- Getting i_2, turn off 3-A source, 12-V source



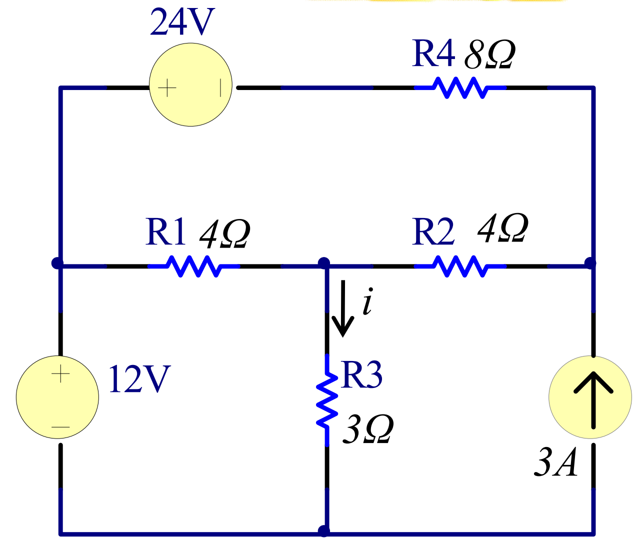


$$i_{R_4} = \frac{24}{(R_1 // R_3) + R_2 + R_4} = 1,75A$$

$$i_2 = \frac{R_1 i}{R_1 + R_3} = -1A$$

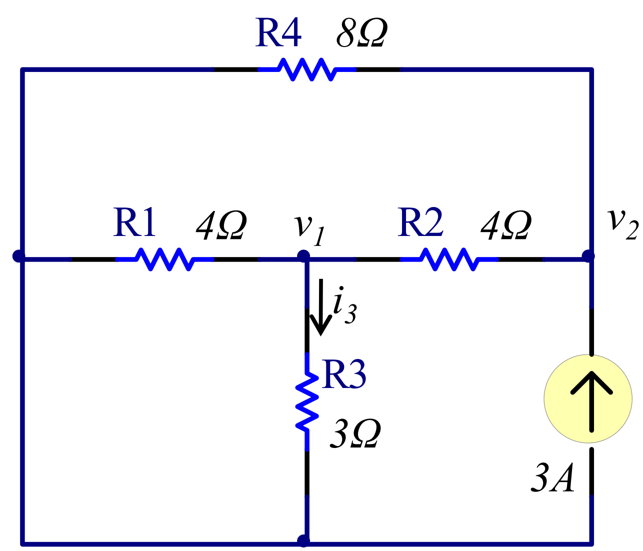
III. Superposition

Ex 4.4: Using the superposition theorem, find *i*.



- To get i_3, turn off 24-V and 12-V sources
- Using nodal analysis:

$$\begin{cases} \left(\frac{1}{8}+\frac{1}{4}\right)v_2 - \frac{1}{4}v_1 = 3 \\ \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{3}\right)v_1 - \frac{1}{4}v_2 = 0 \end{cases} \rightarrow \begin{cases} -2v_1 + 3v_2 = 24 \\ v_2 = 3,33v_1 \end{cases}$$



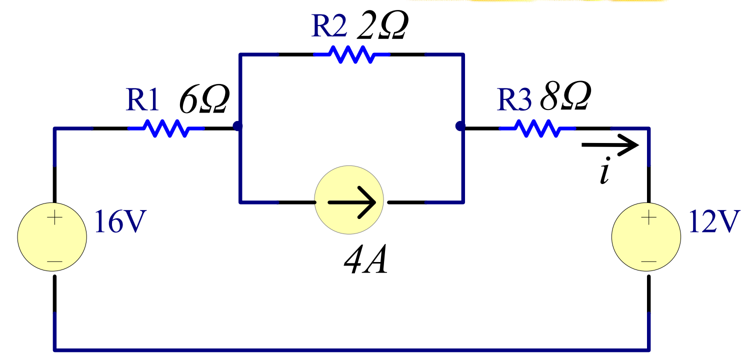
- Solving the set of equations gives:

$$v_1 = 3V \rightarrow i_3 = 1A$$

- Thus: $i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2A$

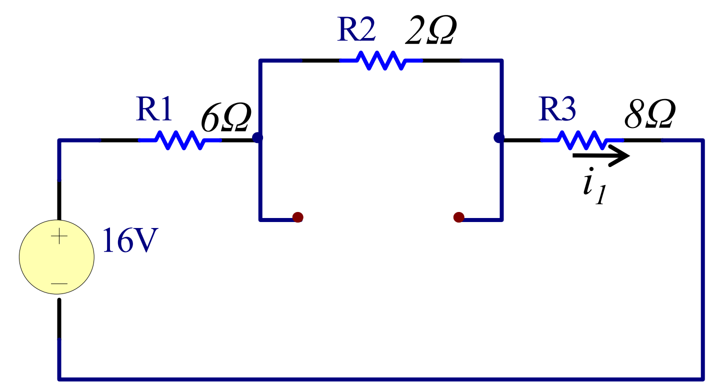
III. Superposition

Ex 4.5: Using the superposition theorem, find *i*.



- The circuit has three sources, let: $i = i_1 + i_2 + i_3$
- Getting i_1, turn off 4-A source, 12-V source

$$i_1 = \frac{16}{6+2+8} = 1A$$

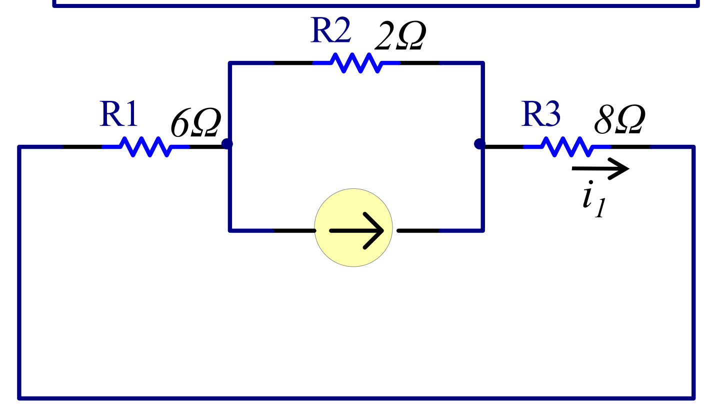


- Getting i_2, turn off 16-V, 12-V source

$$i_2 = \frac{4}{6+2+8}2 = 0,5A$$

- Getting i_3, turn off 16-V, 4-A source

$$i_3 = \frac{-12}{6+2+8} = -0,75A$$



- Thus: $i = i_1 + i_2 + i_3 = 0,75A$

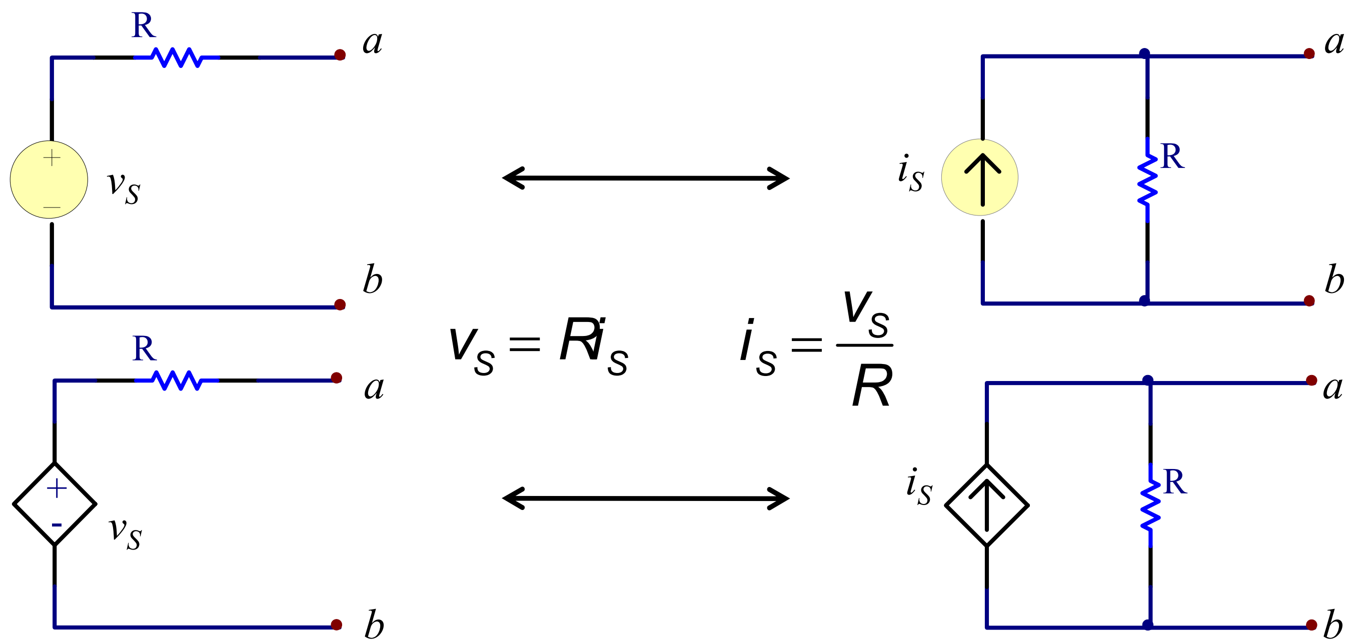
IV. Source transformation

- Similar to the series-parallel combination and wye-delta transformation, ***source transformation*** is using to simplify circuits that bases on the ***concept of equivalence***.

- ***An equivalent circuit*** is one whose v - i characteristics are identical with the original circuit.

A ***source transformation*** is the process of replacing a voltage source v_s in series with a resistor *R* by a current source i_s in parallel with a resistor R, or vice versa.

IV. Source transformation

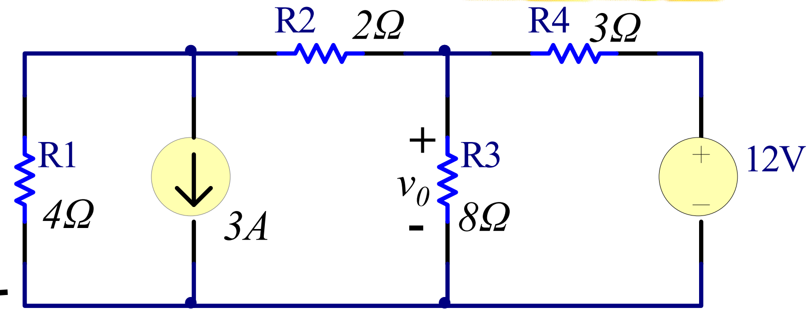


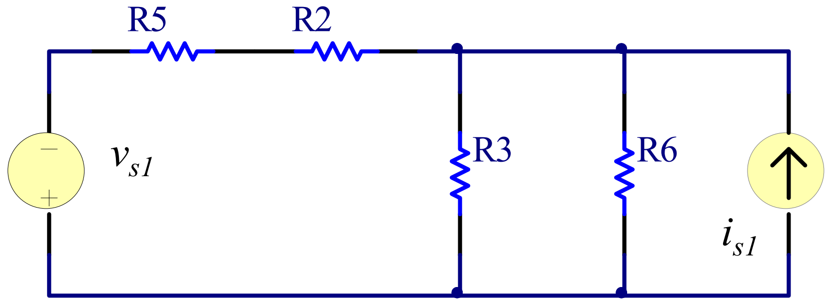
$$v_S = Ri_S \qquad i_S = \frac{v_S}{R}$$

- ***Notes:***
  - The *arrow of the current source is directed toward the positive terminal of the voltage source*.
  - The source transformation is not possible when *R = 0* (*ideal voltage source*) or *R = ∞* (*ideal current source*).

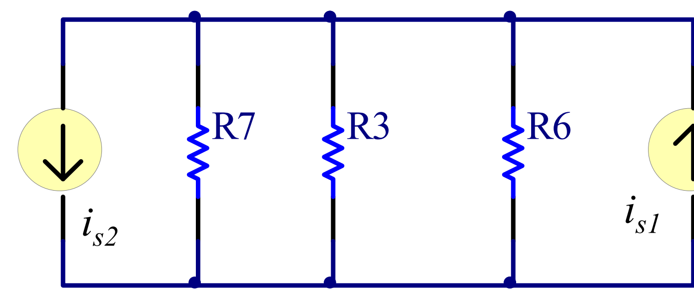
IV. Source transformation

Ex 4.6: Use source transformation to find v_0

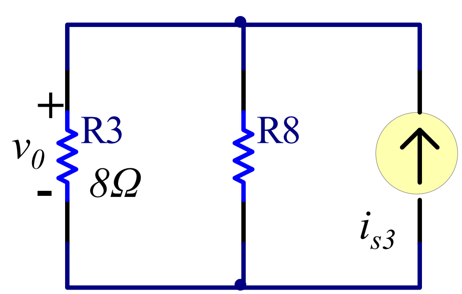




$$\begin{cases} v_{S1} = 3R_1 = 12V \\ R_5 = R_1 = 4\Omega \end{cases} ; \quad \begin{cases} i_{S1} = \dfrac{12}{R_4} = 4A \\ R_6 = R_4 = 3\Omega \end{cases}$$



$$\begin{cases} R_7 = R_2 + R_5 = 6\Omega \\ i_{S2} = \dfrac{v_{S1}}{R_7} = 2A \end{cases}$$

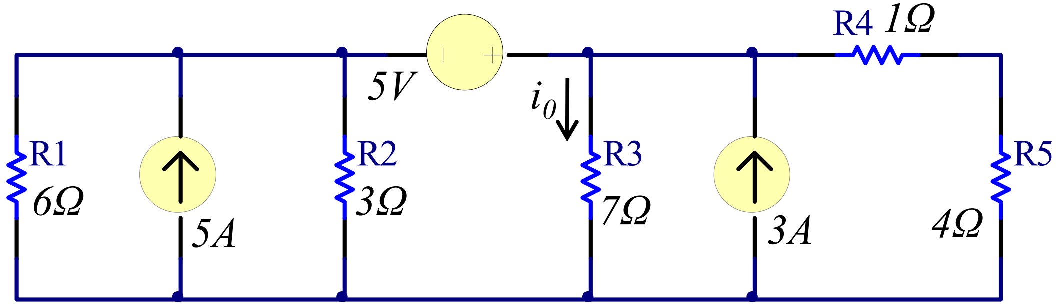


$$\begin{cases} R_8 = R_7 // R_6 = 2\Omega \\ i_{S3} = i_{S1} - i_{S2} = 2A \end{cases}$$

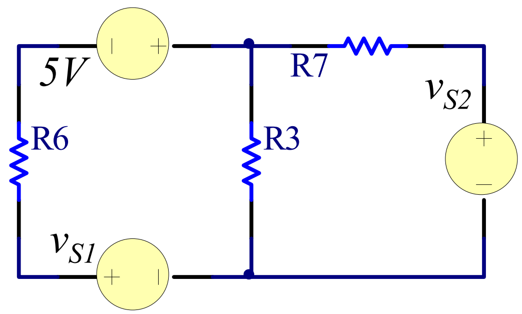
$$\rightarrow v_0 = i_{S3} \frac{R_8 . R_3}{R_3 + R_8} = 2 \frac{2.8}{10} = 3,2V$$

IV. Source transformation

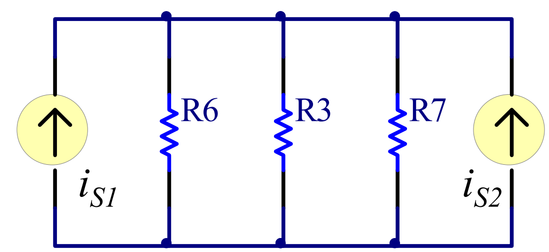
Ex 4.7: Use source transformation to find i_0



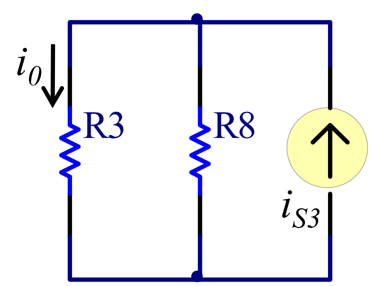
$$\begin{cases} R_6 = R_1 // R_2 = 2\Omega \\ v_{S1} = 5R_6 = 10V \end{cases} ; \quad \begin{cases} R_7 = R_4 + R_5 = 5\Omega \\ v_{S2} = 3R_7 = 15V \end{cases}$$



$$\rightarrow \begin{cases} i_{S1} = \dfrac{v_{S1} + 5}{R_6} = 7,5A \\ i_{S2} = \dfrac{v_{S2}}{R_7} = 3A \end{cases} \rightarrow \begin{cases} i_{S3} = i_{S1} + i_{S2} = 10,5A \\ R_8 = \dfrac{R_6 R_7}{R_6 + R_7} = 1,43\Omega \end{cases}$$

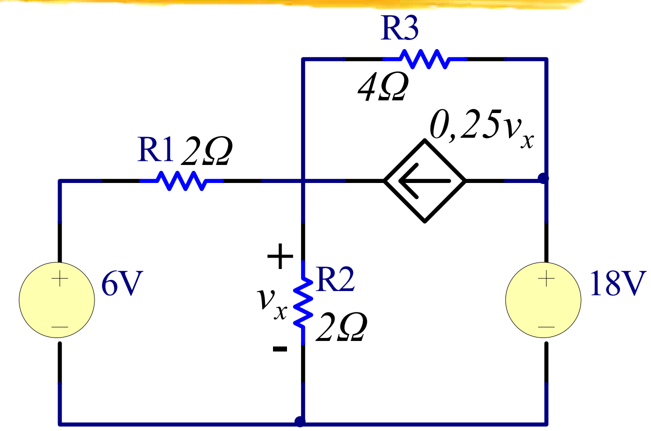


$$\rightarrow i_0 = i_{S3} \frac{R_8}{R_8 + R_3} = 1,78A$$

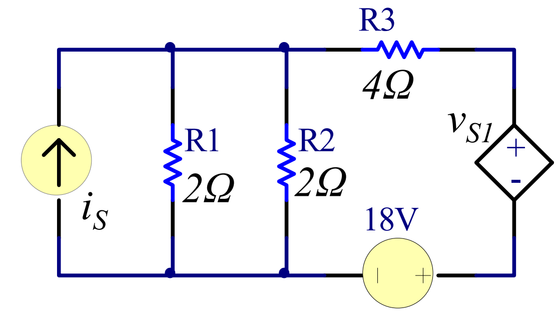


IV. Source transformation

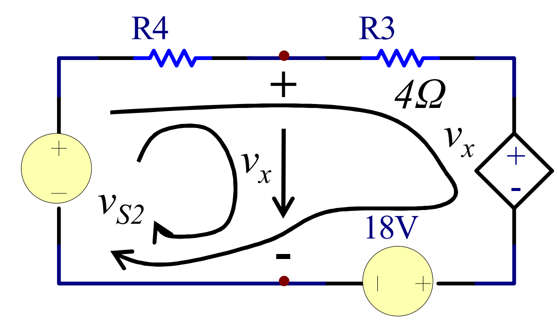
Ex 4.8: Use source transformation to find v_x



- We transform:
  - 6-V independent voltage source: $i_S = \frac{6}{2} = 3A$
  - Dependent current source: $v_{S1} = 0,25v_x.R_3 = v_x$
  - i_S independent current source and R_1, R_2



$$\begin{cases} R_4 = R_1 // R_2 = 1\Omega \\ v_{S2} = i_S R_4 = 3V \end{cases}$$

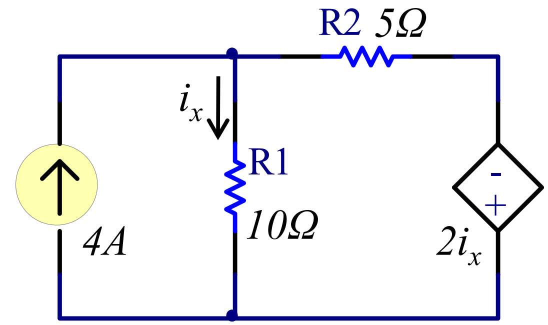


- Applying KVL:
  - The largest loop: $-3 + 5i + v_x + 18 = 0$
  - Loop containing v_{S2} and R_4: $-3 + i + v_x = 0$

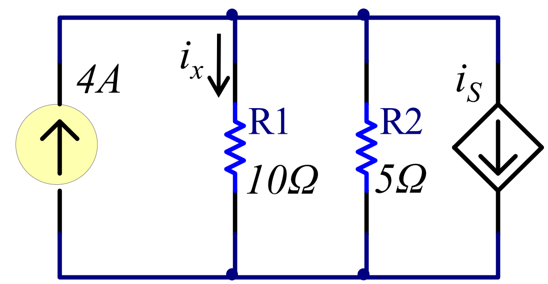
$$\rightarrow \begin{cases} i = -4,5A \\ v_x = 7,5V \end{cases}$$

IV. Source transformation

Ex 4.9: Use source transformation to find i_x



- We transform:
  - Dependent voltage source: $i_S = \dfrac{2i_x}{5} = 0,4i_x$

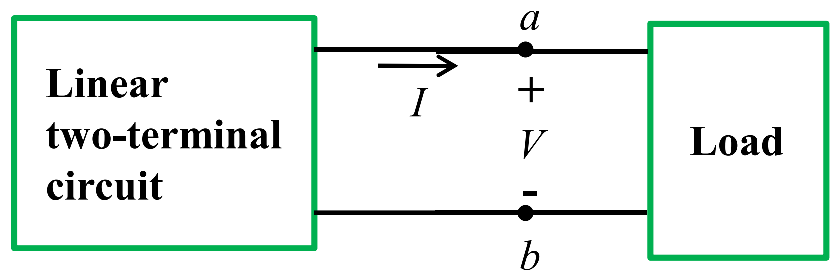


- Applying KCL gives:

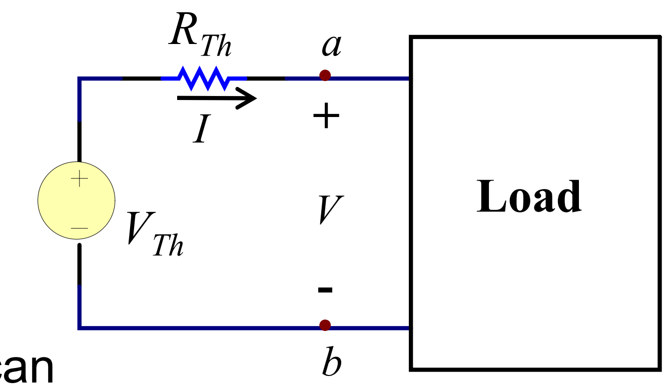
$$i_x = \left(4 - 0,4i_x\right)\frac{R_2}{R_1 + R_2} = \frac{4}{3} - \frac{0,4i_x}{3}$$

$$\rightarrow 3,4i_x = \frac{4}{3} \rightarrow i_x = 0,39A$$

V. Thevenin's theorem



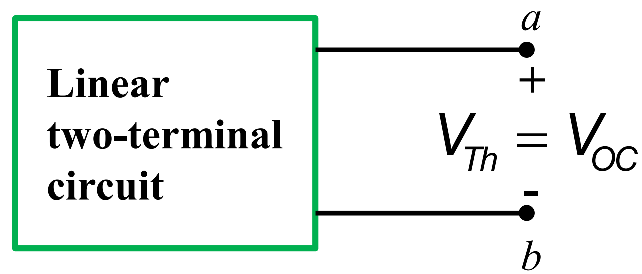
- In practice, a particular element in a circuit is variable (called *load*) while other elements are fixed.



- Each time the variable element is changed, need to be analyzed all over again → use Thevenin's theorem to avoid this problem.
- ***Thevenin's theorem:*** A linear two terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resister R_{Th} where:
  - ❖ V_{Th} is the ***open-circuit voltage*** at the terminals
  - ❖ R_{Th} is the ***input or equivalent resistance*** at the terminals when the independent sources are turned off.

V. Thevenin's theorem

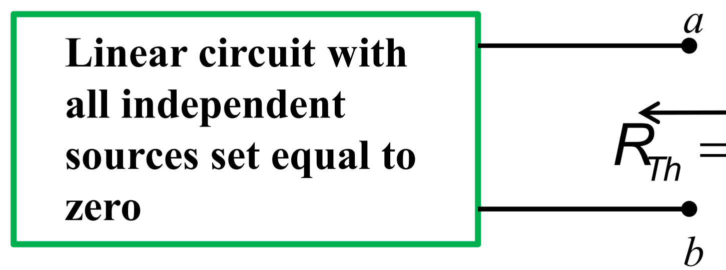
➢ Finding V_{Th}: V_{Th} is the open-circuit voltage across the terminals.



$$V_{Th} = V_{OC}$$

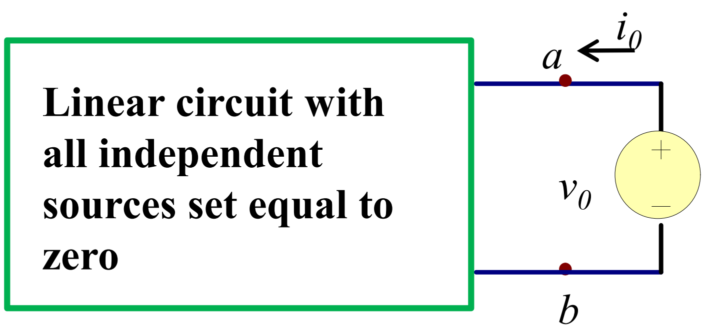
➢ Finding R_{Th}:

❖ Network has no dependent sources.

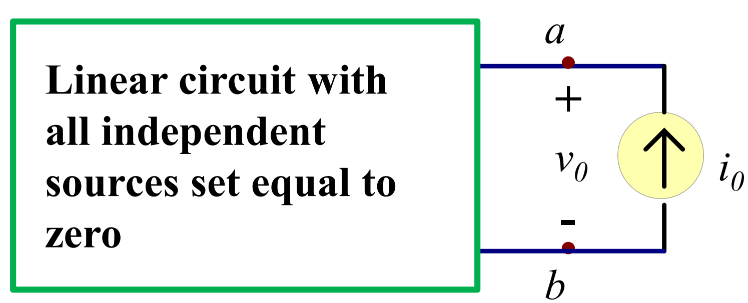


$$R_{Th} = R_{in}$$

❖ Network has dependent sources.



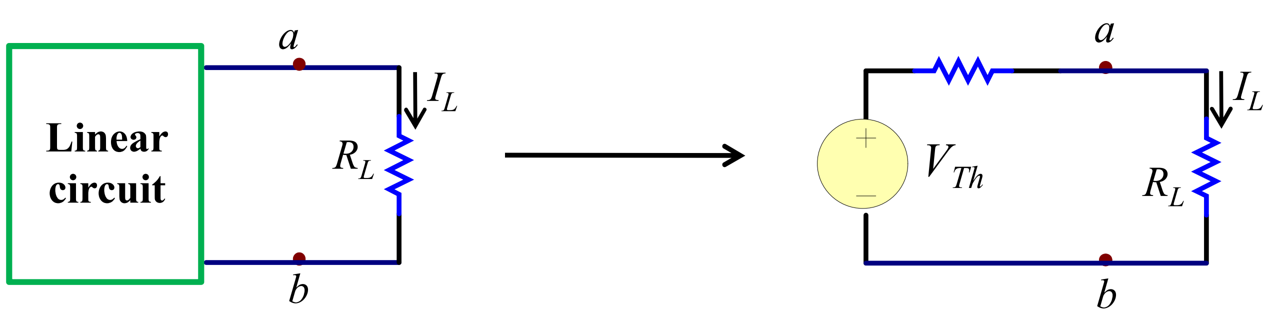
$$R_{Th} = \frac{v_0}{i_0}$$



$$R_{Th} = \frac{v_0}{i_0}$$

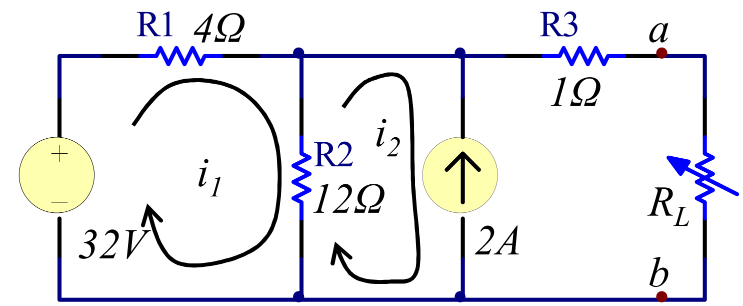
V. Thevenin's theorem

- Thevenin's theorem is very important in circuit analysis:
  - Help simplify a circuit: Replace a large circuit by a single independent voltage source and a single resistor.
  - Easily to determine the current and voltage on the load



$$I_L = \frac{V_{Th}}{R_{Th} + R_L} \qquad V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th}$$

V. Thevenin's theorem



Ex 4.10: Find the Thevenin equivalent circuit of the circuit. Find the current through R_L = *6, 16, 36Ω*

➢ Calculating R_{Th}: $R_{Th} = \dfrac{R_1 R_2}{R_1 + R_2} + R_3 = 4\Omega$

➢ Calculating V_{Th}: Applying nodal analysis gives $\dfrac{32 - V_{Th}}{R_1} + 2 = \dfrac{V_{Th}}{12} \rightarrow V_{Th} = 30V$

➢ The current through R_L : $I_L = \dfrac{V_{Th}}{R_{Th} + R_L} = \dfrac{30}{4 + R_L}$

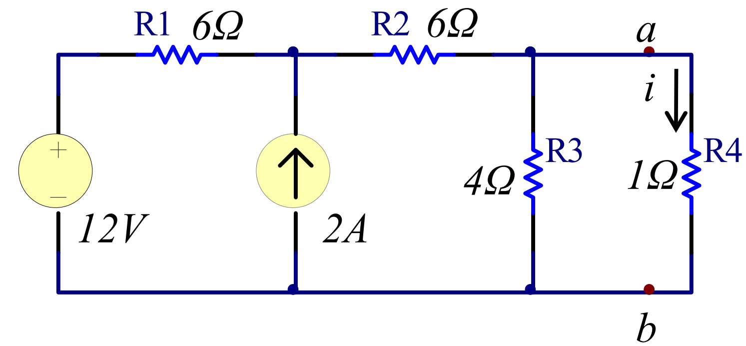
❖ When R_L = *6Ω:* $I_L = \dfrac{30}{10} = 3A$

❖ When R_L = *36Ω:* $I_L = \dfrac{30}{40} = 0,75A$

❖ When R_L = *16Ω:* $I_L = \dfrac{30}{20} = 1,5A$

V. Thevenin's theorem

Ex 4.11: Find *i* by using the Thevenin's theorem



- Calculating R_{Th}:

$$R_{Th} = (R_1 + R_2)//R_3 = \frac{12.4}{12+4} = 3\Omega$$

- Calculating V_{Th}: Applying nodal analysis gives:

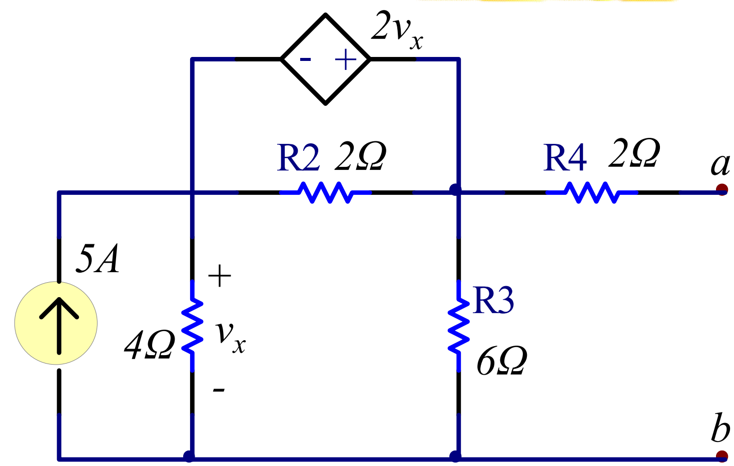
$$\frac{12-V}{6} + 2 = \frac{V}{10} \rightarrow V = 15V \qquad \rightarrow V_{Th} = \frac{V}{R_2 + R_3} R_3 = \frac{15}{6+4}.4 = 6V$$

- The current through R_4 :

$$i = \frac{V_{Th}}{R_{Th} + R_4} = \frac{6}{1+3} = 1,5A$$

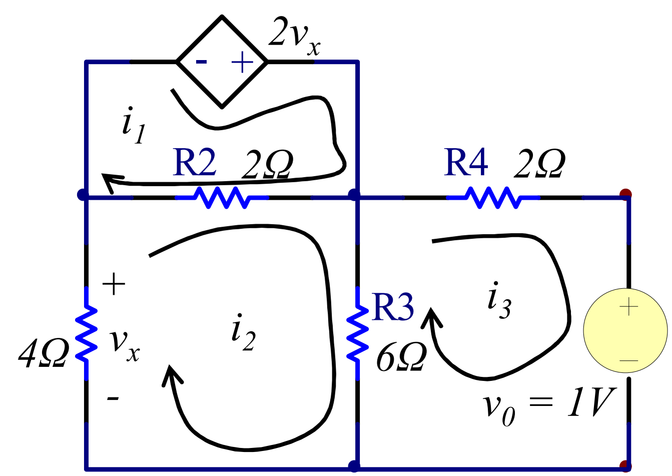
V. Thevenin's theorem

Ex 4.12: Find the equivalent of the circuit



- To find R_{Th}, set the independent source equal to zero, but leave the dependent source alone
- Connect to the terminal a voltage source v_0 = 1V, and we find i_0 through the terminal.

$$R_{Th} = \frac{v_0}{i_0} = \frac{1}{i_0}$$



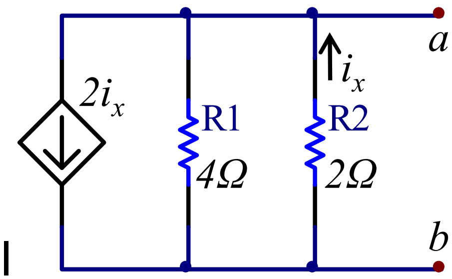
- Applying mesh analysis to loop 1, 2, 3:

- The current through R_4 :

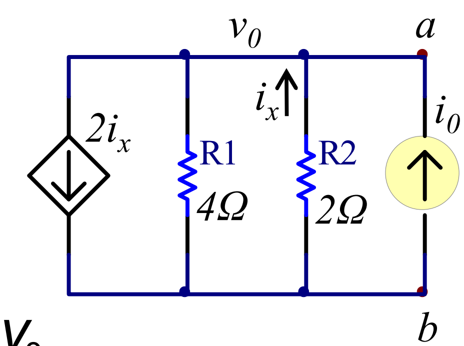
$$\begin{cases} -2v_x + 2(i_1 - i_2) = 0 \\ 4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0 \\ 6(i_3 - i_2) + 2i_3 + 1 = 0 \end{cases} \qquad \rightarrow i_0 = \frac{1}{6}A \qquad \rightarrow R_{Th} = \frac{1}{i_0} = 6\Omega$$

V. Thevenin's theorem

Ex 4.12: Find the equivalent of the circuit



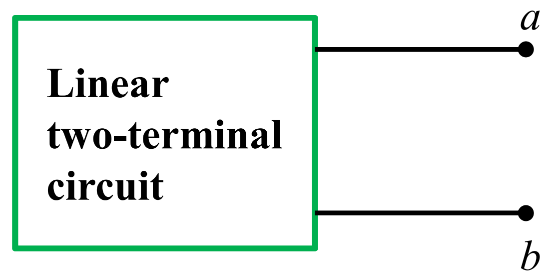
- Since the circuit has no independent sources → $V_{Th} = 0$
- In order to find R_{Th} , apply a current source i_0 at the terminal
- Applying nodal analysis gives: $i_0 + i_x = 2i_x + \dfrac{v_0}{4}$
- Applying Ohm's law: $i_x = \dfrac{0 - v_0}{2} = -\dfrac{v_0}{2}$
- From these two equations, we have:



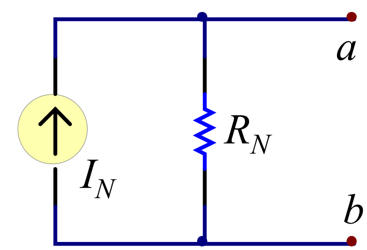
$$i_0 = i_x + \frac{v_0}{4} = -\frac{v_0}{2} + \frac{v_0}{4} = -\frac{v_0}{4} \rightarrow v_0 = -4i_0 \rightarrow R_{Th} = \frac{v_0}{i_0} = -4\Omega$$

- **Note that:**
  - The negative value of R_{Th} means that the circuit is supplying power by the dependent source.
  - This example shows how a dependent source and resistors could be used to simulated negative resistance.

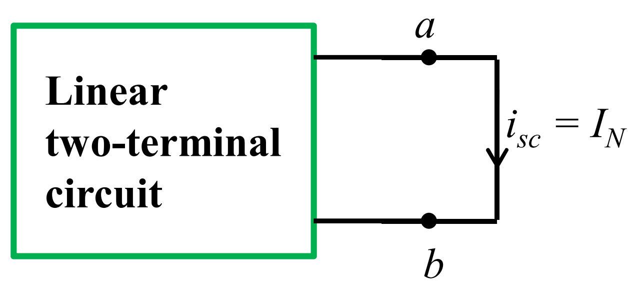
VI. Norton's theorem



- ***Norton's theorem:*** A linear two terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where:
  - ❖ I_N is the short circuit current through the terminals
  - ❖ R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.



- Finding R_N: $R_N = R_{Th}$
- Finding I_N: $I_N = i_{sc}$



- Source transformation: Relationship between Norton's and Thevenin's theorems:

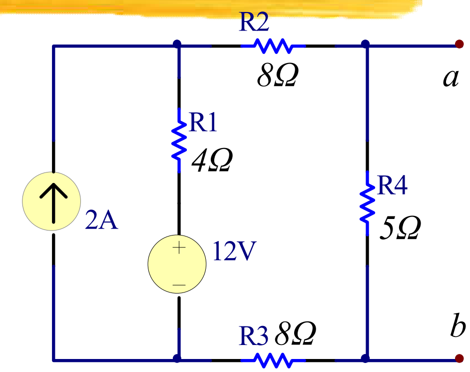
$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}}$$

VI. Norton's theorem

- In order to determine the Thevenin or Norton equivalent circuit, we need to find:
  - The open-circuit voltage v_{oc} across terminals *a* and *b*
  - The short-circuit current i_{sc} at terminals *a* and *b*
  - The equivalent or input resistance R_{in} at terminals *a* and *b* when all independent sources are turned off.

$$V_{Th} = v_{oc} \quad ; \quad I_N = i_{sc} \quad ; \quad R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N$$

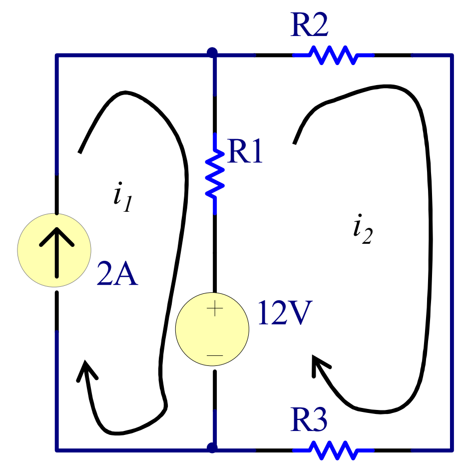
VI. Norton's theorem



Ex 4.13: Find the Norton equivalent circuit for the circuit.

- Finding R_N in the same way R_{Th}

$$R_N = (R_1 + R_2 + R_3) // R_4 = \frac{5(4+8+8)}{5+4+8+8} = 4\Omega$$



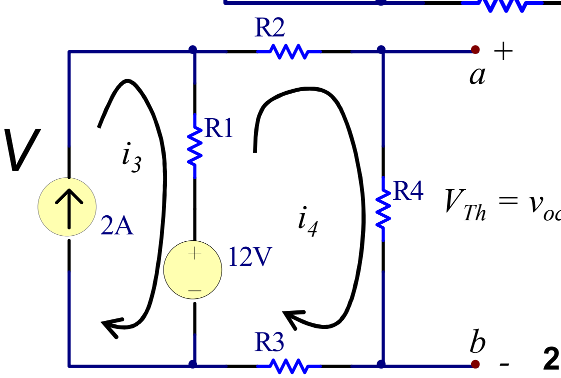
- Finding I_N by shortening circuit terminals *a* and *b*

$$\begin{cases} i_1 = 2A \\ 20i_2 - 4i_1 - 12 = 0 \end{cases} \rightarrow i_2 = i_{sc} = I_N = 1A$$

- By another way, we can find IN by the source transform equation:

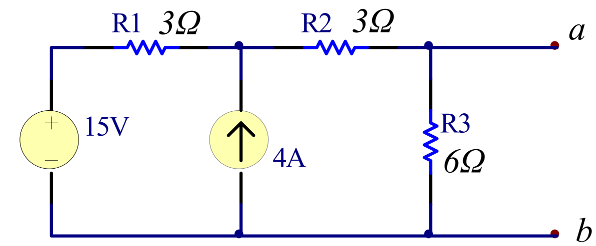
$$\begin{cases} i_3 = 2A \\ 25i_4 - 4i_3 - 12 = 0 \end{cases} \rightarrow i_4 = 0,8A \rightarrow v_{oc} = V_{Th} = 5i_4 = 4V$$

$$\rightarrow I_N = \frac{V_{Th}}{R_N} = 1A$$



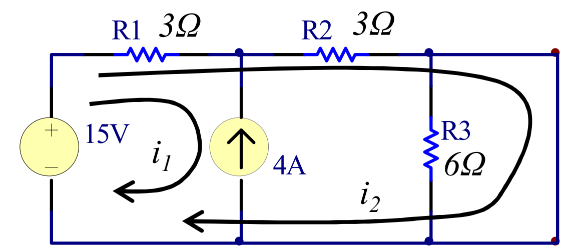
VI. Norton’s theorem

Ex 4.14: Find the Norton equivalent circuit for the circuit.



- Finding R_N

$$R_N = (R_1 + R_2)//R_3 = \frac{6.6}{6+6} = 3\Omega$$

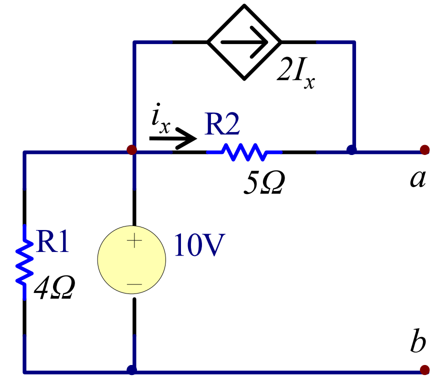


- Finding I_N by shortening circuit terminals *a* and *b* → *applying the mesh analysis gives:*

$$\begin{cases} i_1 = -4A \\ 3(i_1 + i_2) + 3i_2 - 15 = 0 \end{cases} \rightarrow 6i_2 = 27 \rightarrow i_2 = i_{sc} = I_N = \frac{27}{6} = 4{,}5A$$

VI. Norton's theorem

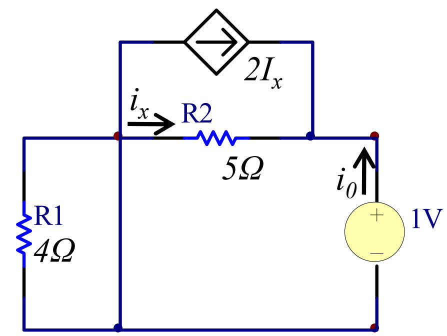
Ex 4.15: Using Norton's theorem, find R_N and I_N at terminals *a-b*



- Finding R_N: set the independent voltage source equal to zero and connect a voltage source of $v_0 = 1V$ to *a-b*

$$i_x = \frac{-v_0}{5} = 0,2A \rightarrow -i_0 = i_x + 2i_x = 3i_x = -0,6A$$

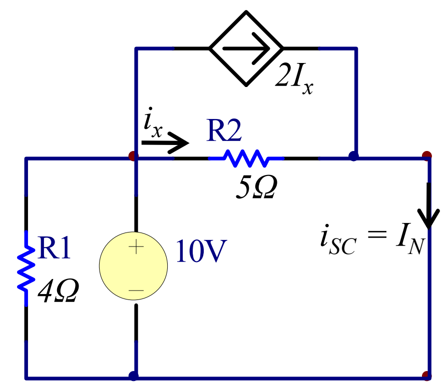
$$\rightarrow R_N = \frac{v_0}{-i_0} = \frac{1}{-0,6} = -1,67\Omega$$



- Finding I_N: Shorting – circuit terminals *a* and *b*

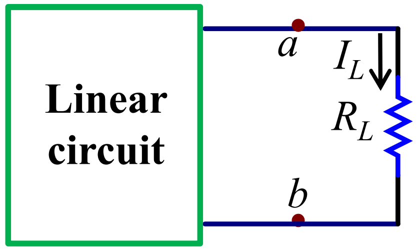
$$i_x = \frac{10}{R_2} = \frac{10}{5} = 2A \qquad \rightarrow I_N = i_{SC} = 6A$$

$$i_{SC} = i_x + 2i_x = 2 + 4 = 6A$$

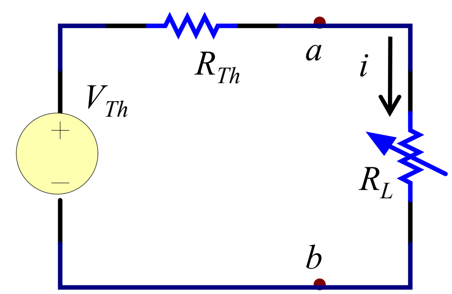


VII. Maximum power transfer

- In many practical situations, a circuit is designed to provide power to a load:
  - Electric utilities: Minimizing power losses in the process distribution
  - Communications: Maximize the power delivered to a load.



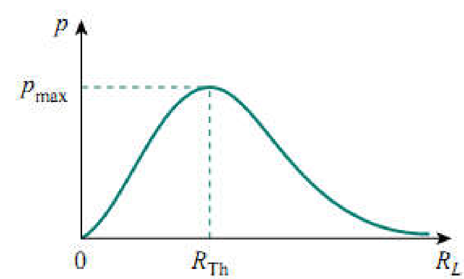
- Problem: Delivering p_{max} to a load when given a system with known internal losses.
  - Assuming that the load resistance R_L can be adjusted
  - Replacing entire circuit by Thevenin equivalent circuit



$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L \xrightarrow{R_L = R_{Th}} p = p_{max}$$

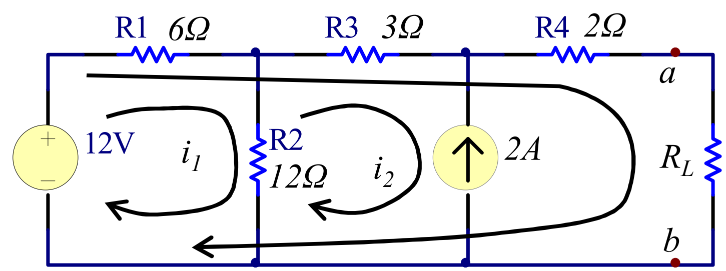
Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$



VII. Maximum power transfer

Ex 4.16: Finding the value of R_L for maximum power transfer. Find the maximum power.



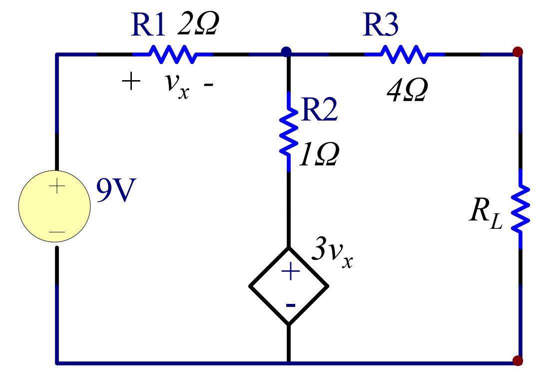
- Finding R_{Th}: $R_{Th} = \left(R_1 // R_2\right) + R_3 + R_4 = 9\Omega$
- Finding V_{Th}:

$$\begin{cases} 6i_1 + 12(i_1 - i_2) = 12 \\ i_2 = -2A \end{cases} \rightarrow \begin{cases} i_1 = \dfrac{-2}{3}A \\ i_2 = -2A \end{cases}$$

- Applying KVL around the outer loop to get V_{Th}: $6i_1 + 3i_2 + V_{Th} = 12 \rightarrow V_{Th} = 22V$
- For maximum power transfer: $R_L = R_{Th} = 9\Omega$
- The maximum power is: $p_{max} = \dfrac{V_{Th}^2}{4.R_L} = \dfrac{22^2}{4.9} = 13,44W$

VII. Maximum power transfer

Ex 4.17: Finding the value of R_L for maximum power transfer. Find the maximum power.



- Finding R_{Th}:
- Finding V_{Th}: