Chapter 6: Capacitors and Inductors

I. Introduction.

II. Capacitors.

III. Series and parallel capacitors.

VI. Inductors.

V. Series and parallel inductors.

VI. Applications.

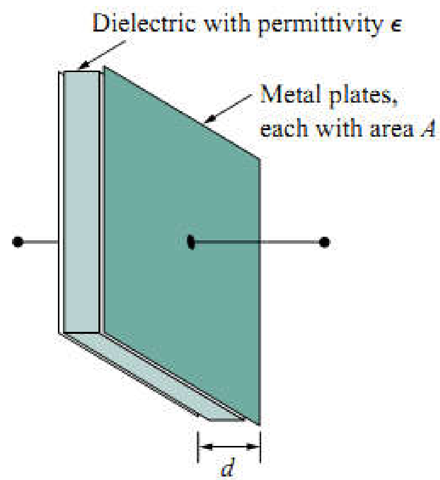
I. Introduction

- So far we have limited our study to *resistive circuits*. In this chapter, we shall introduce two new and important passive linear circuit elements:
  - The capacitor
  - The inductors
- Capacitors and inductors *do not dissipate but store energy* which can be retrieved at a later time → be called *storage elements.*
- With the introduction of capacitors and inductors, we will be able to analyze more important and practical circuits (applying the circuit analysis techniques)
- Applications: Capacitors can be combined with op amps to form *integrators, differentiators, and analog computers.*

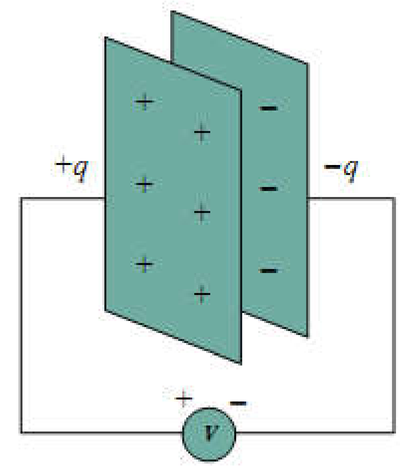
II. Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- A *capacitor* consists of two conducting plates separated by an insulators (or dielectric) (*air, ceramic, paper, mica*).
- When a voltage v is connected:
  - Positive charge q on one plate
  - Negative charge $-q$ on the other

$$\boxed{q = Cv} \qquad 1F = \frac{1C}{1V}$$



A typical capacitor



A capacitor with applied voltage v

- *Capacitance* is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (*F*)

II. Capacitors

- Capacitance C depend on its physical dimension (not *q* and *v*)

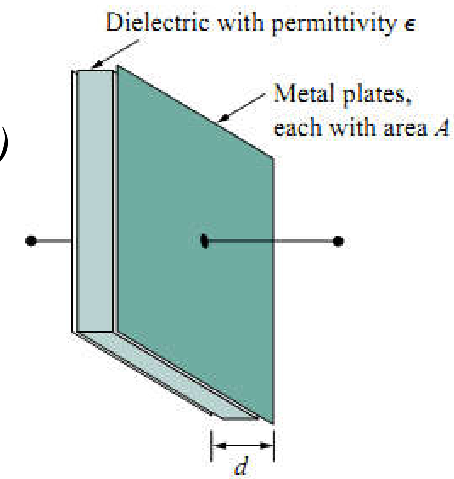
$$\boxed{C = \frac{\varepsilon A}{d}}$$

Apply only for parallel plate capacitors

A: surface area of each plate

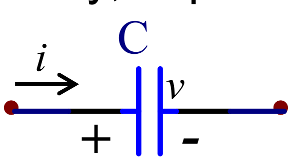
d: distance between the plates

: the permittivity of the dielectric material between the plates

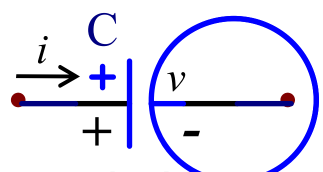


A typical capacitor

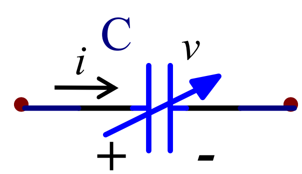
- Typically, capacitors have values in the *pF* to μF



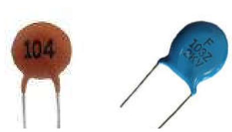
Fix, un-polarized capacitor



Fix, polarized capacitor



Variable capacitor



Ceramic capacitor

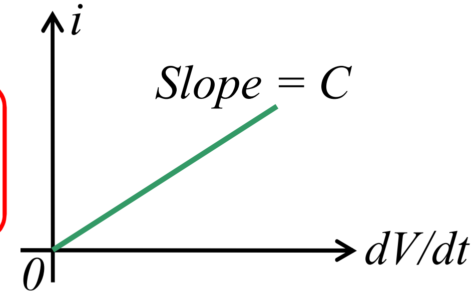
Electrolytic capacitor

Trimmer capacitor

II. Capacitors

- Capacitors are used to block dc, pass ac, shift phase, store energy, start motors and suppress noise.

$$\boxed{i = \frac{dq}{dt} = \frac{\partial q}{\partial u}.\frac{dv}{dt} = C\frac{dv}{dt}} \qquad \boxed{v = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)}$$

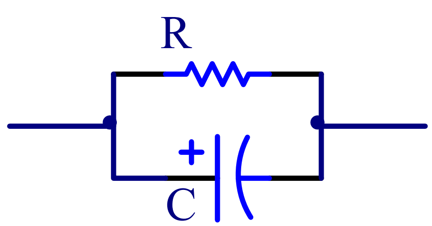


Current-voltage relationship

- Classification:
  - *Linear capacitor* ← We focus only the linear capacitor
  - *Nonlinear capacitor* (a few)
- Instantaneous power: $p = vi = Cv\frac{dv}{dt}$
- Energy: $w = \int_{-\infty}^{t} p.dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = \frac{1}{2}Cv^2\Big|_{t=-\infty}^{t} \rightarrow \boxed{w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}}$

II. Capacitors

- Properties of a capacitor:
  - A capacitor is an *open circuit to DC.*
  - The voltage on a capacitor *cannot change abruptly:* An discontinuous change in voltage requires an infinitive current.
  - The ideal capacitor does *not dissipate energy*: It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
  - A *non-ideal capacitor* has a parallel-model leakage resistance (~100MΩ the leakage resistance may be neglected for most practical application).



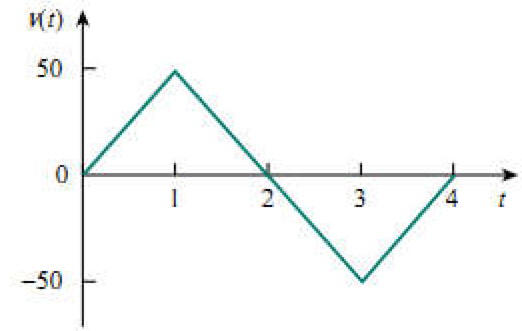
Circuit model of a non-ideal capacitor

II. Capacitors

Ex 6.1: Determine the current through a 200μF capacitor whose voltage is

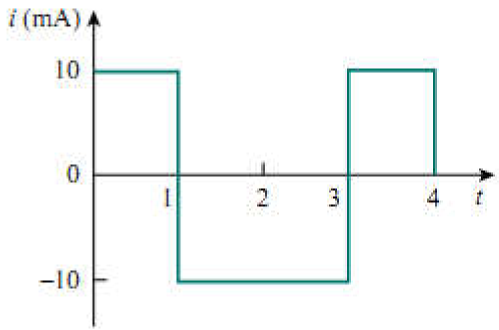
- The voltage can be described mathematically as:

$$v(t)=\begin{cases} 50t\ V & 0<t<1 \\ 100-50t\ V & 1<t<3 \\ -200+50t\ V & 3<t<4 \\ 0\ V & otherwise \end{cases}$$



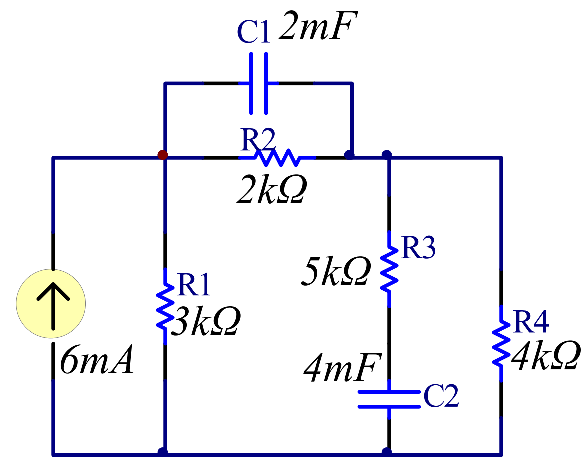
- Applying $i = C\,dv/dt$ gives

$$i(t)=200.10^{-6}.\begin{cases} 50\ V & 0<t<1 \\ -50\ V & 1<t<3 \\ 50\ V & 3<t<4 \\ 0\ V & otherwise \end{cases}=\begin{cases} 10\ mA & 0<t<1 \\ -10\ mA & 1<t<3 \\ 10\ mA & 3<t<4 \\ 0\ mA & otherwise \end{cases}$$



II. Capacitors

Ex 6.2: Obtain the energy stored in each capacitor under DC condition



- Under DC conditions, each capacitor is an open circuit
- Applying the current division gives

$$i = \frac{R_1}{R_1 + R_2 + R_4} 6mA = 2mA$$

- The voltages across C_1 and C_2 are

$$v_1 = R_2.i = 2000.2.10^{-3} = 4V \qquad v_2 = R_4.i = 4000.2.10^{-3} = 8V$$

- The energies stored in C_1, C_2

$$w_1 = \frac{1}{2}C_1v_1^2 = 16mJ \quad ; \quad w_2 = \frac{1}{2}C_2v_2^2 = 128mJ$$

III. Series and parallel capacitors

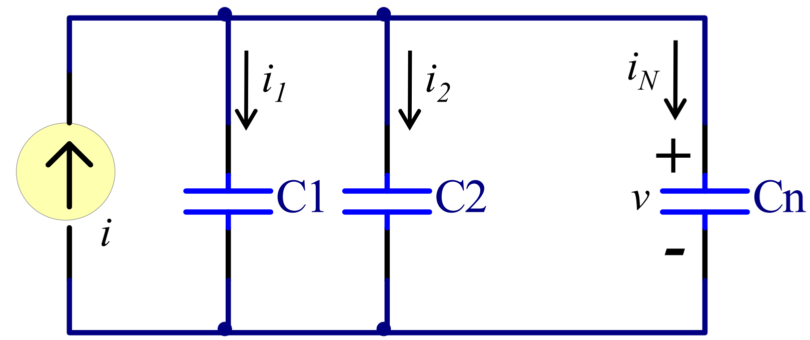
➢ In order to obtain the equivalent capacitor C_{eq} of N capacitor in parallel, we applying KCL

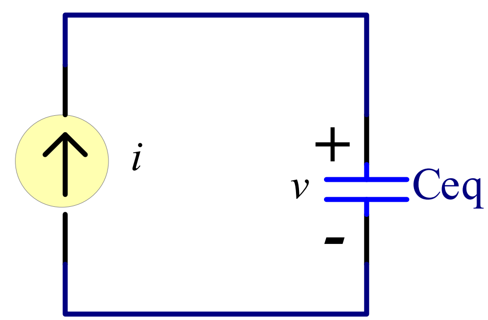
$$i = i_1 + i_2 + i_3 + \ldots + i_N$$

$$\rightarrow i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt}$$

$$\rightarrow i = \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

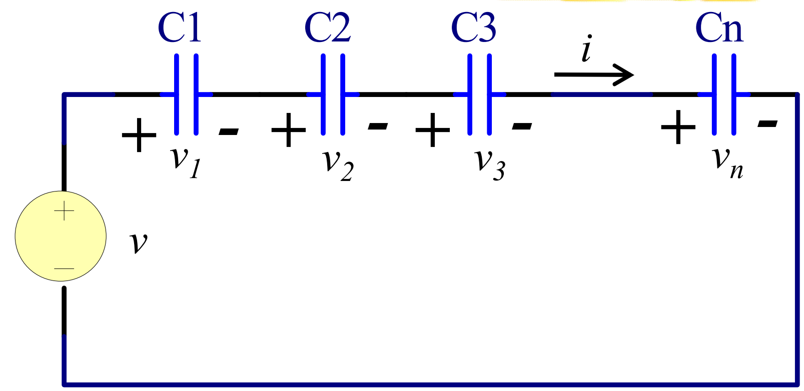
$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \ldots + C_N}$$





➢ The *equivalent capacitance* of N parallel-connected capacitors is the sum of the individual capacitances

III. Series and parallel capacitors



- Applying KVL to the loop:

$$v = v_1 + v_2 + v_3 + \ldots + v_N$$

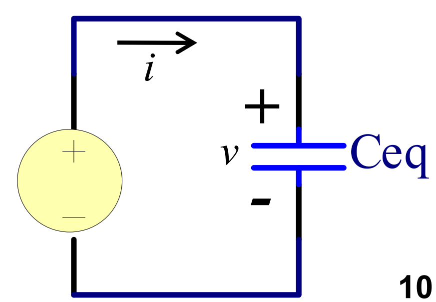
$$But \quad v_k = \frac{1}{C_k}\int_{t_0}^{t} i\,dt + v_k(t_0)$$

$$\rightarrow v = \frac{1}{C_1}\int_{t_0}^{t} i\,dt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i\,dt + v_2(t_0) + \ldots + \frac{1}{C_N}\int_{t_0}^{t} i\,dt + v_N(t_0)$$

$$\rightarrow v = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}\right)\int_{t_0}^{t} i\,dt + v_1(t_0) + v_2(t_0) + \ldots + v_N(t_0) = \frac{1}{C_{eq}}\int_{t_0}^{t} i\,dt + v(t_0)$$

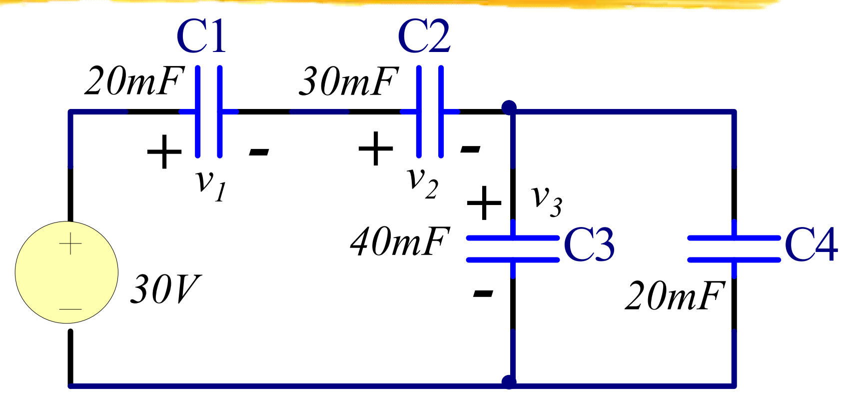
- The *equivalent capacitance* of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}}$$



III. Series and parallel capacitors

Ex 6.3: Find the voltage across each capacitor.



- C_3 and C_4 can be combined to get:

$$C_{34} = 40 + 20 = 60mF$$

- This capacitor is in series with C_1 and C_2

$$C_{eq} = \frac{1}{\frac{1}{C_{34}} + \frac{1}{C_1} + \frac{1}{C_2}} = 10mF$$

- The total charge: $q = C_{eq} v = 10.10^{-3}.30 = 0,3C$ (charge on the C_1 and C_2)

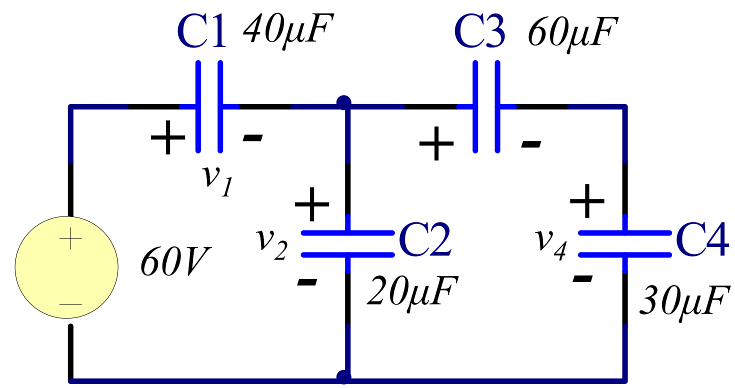
$$\rightarrow v_1 = \frac{q}{C_1} = \frac{0,3}{20.10^{-3}} = 15V \;\; ; \;\; v_2 = \frac{q}{C_2} = \frac{0,3}{30.10^{-3}} = 10V \;\; \rightarrow v_3 = 30 - v_1 - v_2 = 5V$$

- *Another solution:* $C_3 // C_4 \rightarrow C_{34} = 20 + 40 = 60mF$

$$\rightarrow C_{34} \textit{ is in series with } C_1 \textit{ and } C_2 \quad \rightarrow v_3 = \frac{q}{C_{34}} = \frac{0,3}{60.10^{-3}} = 5V$$

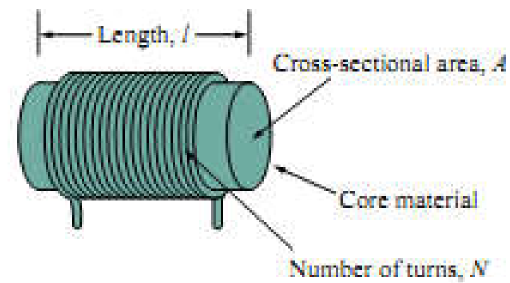
III. Series and parallel capacitors

Ex 6.4: Find the voltage across each capacitor.



IV. Inductors

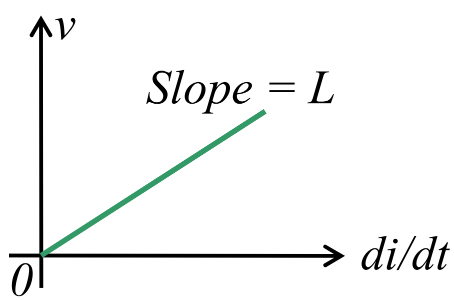
- *An inductor* is a passive element designed to store energy in its magnetic field.
- An inductor consists of a coil of conducting wire.
- If current is allowed to pass through an inductor, the voltage across the inductor is directly proportional to the time rate of change of the current.



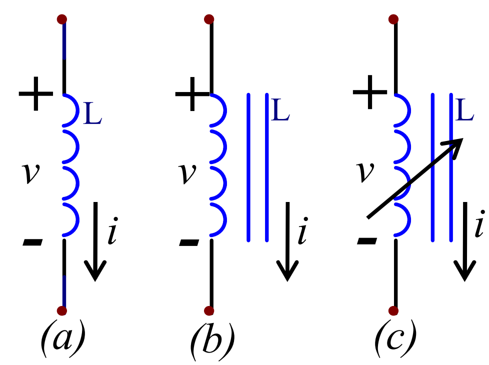
Typical form of an inductor

$$v = L\frac{di}{dt} \quad ; \quad i = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0)$$

L: inductance of the inductor [H]



Voltage-current relationship



Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core

IV. Inductors

- *Inductance* is the property whereby an inductor exhibits opposition to the charge of current flowing through it, measured in [H]. It depends on:
  - Physical dimension
  - Construction

$$\boxed{L = \frac{N^2 \mu A}{l}}$$

N: number of turns

l: length of coil

A: cross-sectional area

μ: permeability of the core

- Inductor classification:
  - Fixed - variable
  - Linear - in-linear
  - Core: iron, steel, plastic, air, …

Toroidal inductor

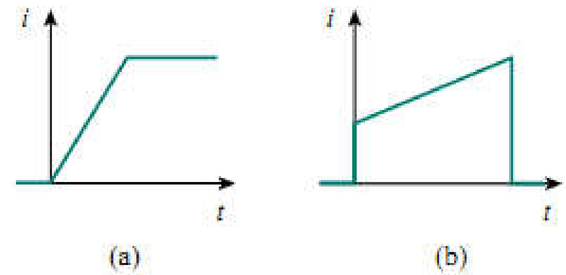
Solenoidal wound inductor

Inductor

IV. Inductors

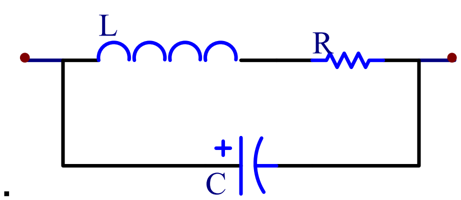
- Power: $p = v.i = \left(L\dfrac{di}{dt}\right)i$
- Energy:

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t}\left(L\frac{di}{dt}\right)i\,dt = L\int_{-\infty}^{t} i\,di \rightarrow \boxed{w = \frac{1}{2}Li^2}$$



Current through an inductor: (a) allowed, (b) not allowable, an abrupt change is not possible.

- Note that:
  - An inductor acts like a *short circuit to DC*
  - The current *cannot change instantaneously*.
  - The ideal inductor does *not dissipate energy.*
  - *Non-ideal inductor* has a significant resistive component:
    - winding resistance (very small)
    - Winding capacitance (very small, except at high frequencies)



Circuit model for a practical inductor

IV. Inductors

Ex 6.5: Find the current through a 5H inductor and the energy stored within 0<t< 5s, if the its voltage is:

$$v(t) = \begin{cases} 30t^2, t > 0 \\ 0, t < 0 \end{cases}$$

- The current through the inductor:

$$i = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0) = \frac{1}{5}\int_{0}^{t} 30t^2 dt + 0 = 2t^3 (A)$$

- The power: $p = v.i = 60t^5$

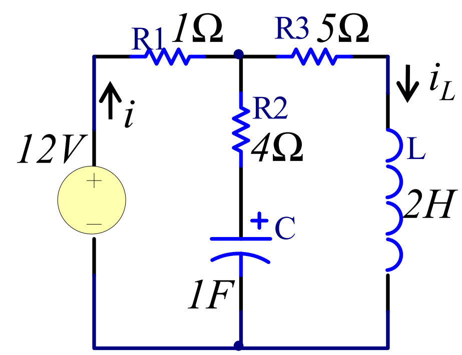
- The energy: $w = \int pdt = \int_0^5 60t^5 dt = 60\left.\frac{t^6}{6}\right|_0^5 = 156,25kJ$

- The energy could be calculated by applying the equation:

$$w = \frac{1}{2}.5.\left(2.5^3\right)^2 = 156,25kJ$$

IV. Inductors

Ex 6.6: Find i, v_C, i_L, energy stored in the capacitor and inductor under DC condition.



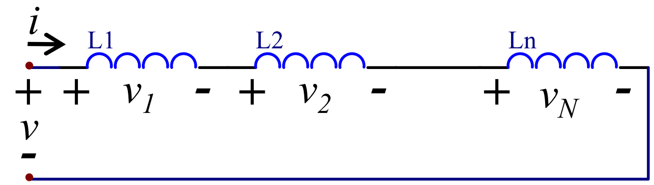
- Under DC condition, replacing:
  - capacitor → open circuit
  - Inductor → short circuit

$$\rightarrow i = i_L = \frac{12V}{R_1 + R_3} = 2A \quad \rightarrow v_C = R_3.i = 5.2 = 10V$$

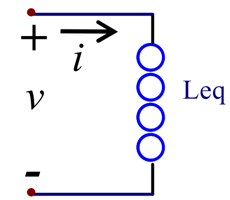
- The energy in the capacitor: $w_C = \frac{1}{2} C v_C^2 = \frac{1}{2}.1.10^2 = 50J$

- The energy in the inductor: $w_L = \frac{1}{2}.L.i_L^2 = \frac{1}{2}.2.2^2 = 4J$

V. Series and parallel inductors



- Consider a series connection of N inductors



  - Applying KVL to the loop:

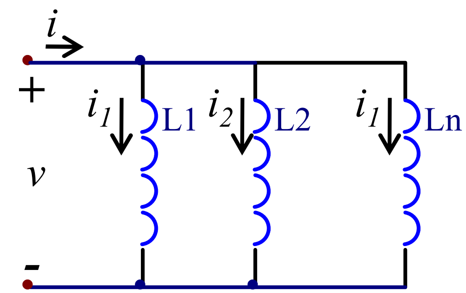
$$v = v_1 + v_2 + v_3 + \ldots + v_N$$

$$\rightarrow v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \ldots + L_N \frac{di}{dt} = \left(\sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

$$\boxed{L_{eq} = L_1 + L_2 + L_3 + \ldots + L_N}$$

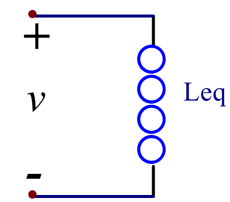
- The equivalent inductance of series-connected inductors is the sum of the individual inductances.

V. Series and parallel inductors



- Consider a parallel connection of N inductors
  - Applying KCL:

$$i = i_1 + i_2 + i_3 + \ldots + i_N$$



$$\rightarrow i = \frac{1}{L_1}\int_{t_0}^{t} vdt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} vdt + i_2(t_0) + \ldots + \frac{1}{L_N}\int_{t_0}^{t} vdt + i_N(t_0)$$

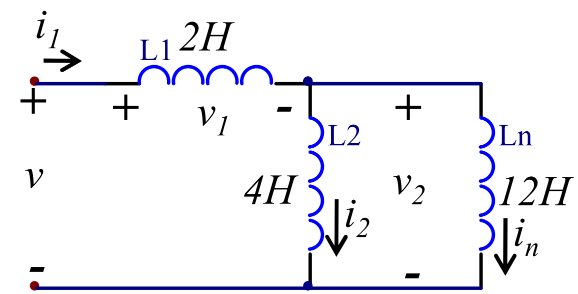
$$\rightarrow i = \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} vdt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} vdt + i(t_0)$$

$$\boxed{L_{eq} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$$

- The equivalent inductance of parallel inductors is reciprocal of the sum of the reciprocals of the individual inductances.

V. Series and parallel inductors

Ex 6.7: Find $i_2(0)$, $v(t)$, $v_1(t)$, $v_2(t)$, $i_2(t)$, $i_n(t)$ if $i_1(t) = 4.(2 - e^{-10t})\ mA$ and $i_n(0) = -1mA$.



- From $i_1(t) = 4.(2 - e^{-10t})\ mA \rightarrow i_1(0) = 4.(2 - 1) = 4mA$

$$i_1 = i_2 + i_n \rightarrow i_2(0) = i_1(0) - i_n(0) = 4 - (-1) = 5mA$$

- The equivalent inductance is

$$L_{eq} = \left(L_2 // L_N\right) + L_1 = 2 + \frac{4.12}{4+12} = 5H$$

- Thus: $v(t) = L_{eq}\dfrac{di_1}{dt} = 5.(-4).(-10)e^{-10t} = 200e^{-10t}mV$

$$v_1(t) = 4\frac{di_1}{dt} = 4.(-4).(-10)e^{-10t} = 80e^{-10t}mV$$

- Thus: $v(t) = v_1 + v_2 \rightarrow v_2(t) = v(t) - v_1(t) = 120e^{-10t}mV$

V. Series and parallel inductors

Ex 6.7: Find $i_2(0)$, $v(t)$, $v_1(t)$, $v_2(t)$, $i_2(t)$, $i_n(t)$ if $i_1(t) = 4.(2 - e^{-10t})\ mA$ and $i_n(0) = -1mA$.

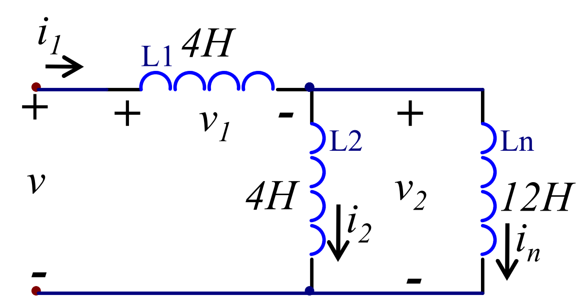
- Current i_1

$$i_2(t) = \frac{1}{4}\int_0^t v_2 dt + i_2(0) = \frac{120}{4}\int_0^t e^{-10t} dt + 5 = 8 - 3e^{-10t}\,(mA)$$

- Current i_n

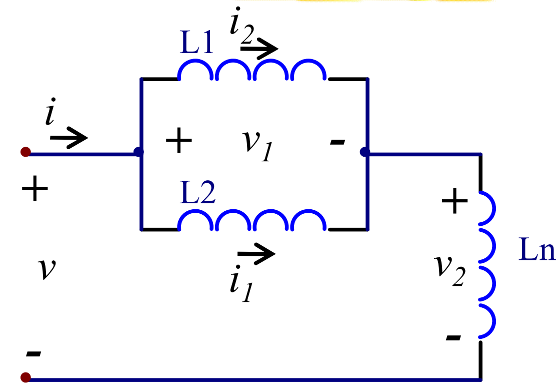
$$i_n(t) = \frac{1}{12}\int_0^t v_2 dt + i_n(0) = \frac{120}{12}\int_0^t e^{-10t} dt - 1 = -e^{-10t}\,(mA)$$

- For validation: $i_1(t) = i_2(t) + i_n(t)$



V. Series and parallel inductors

Ex 6.8: Find $i_2(0)$, $i_2(t)$, $i(t)$, $v(t)$, $v_1(t)$, $v_2(t)$ nếu $i_1(t) = 0,6.e^{-2t}$ và $i(0) = 1,4A$

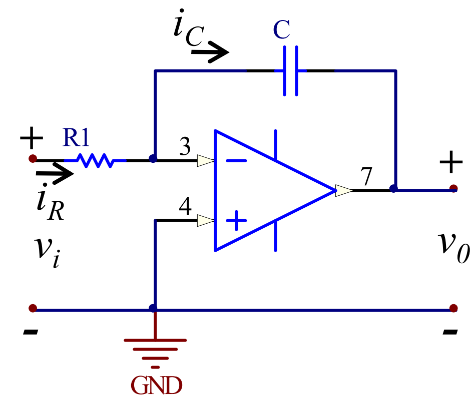


VI. Applications

- Circuit elements (R, C) are available in either discrete form or integrated circuit (IC) form, but inductance are difficult to produce on IC substrates.
- The inductor are used in some applications:
  - Relays, delays, sensing devices, pick-up head
  - Telephone circuits, radio, TV receivers
  - Power supplies, electric motors, microphones, loudspeakers
- Capacitors and inductors possess 03 special properties:
  - Useful for generating a current or voltage in short period of time (DC circuit).
  - Useful for suppression and converting pulsating DC voltage into relatively smooth DC voltage (DC circuit).
  - Useful for frequency discrimination (AC circuit).

VI. Applications

VI.1. Integrator



- An *integrator* is an op amp circuit whose output is proportional to the integral of the input signal.

- At node 3:

$$i_R = i_C \rightarrow \frac{v_i}{R} = i_R = i_C = -C\frac{dv_0}{dt} \rightarrow v_0(t) - v_0(0) = -\frac{1}{RC}\int_0^t v_i(t)dt$$

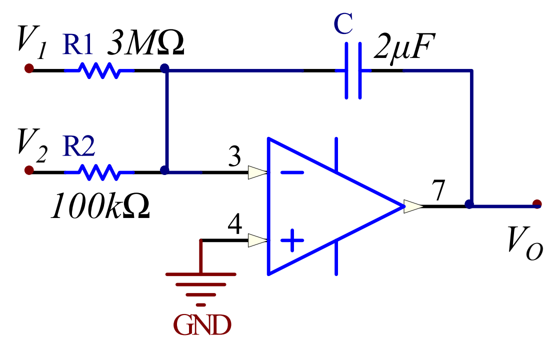
$$\boxed{v_0(t) = -\frac{1}{RC}\int_0^t v_i(t)dt}$$

- In practice, note that:
  - The op amp integrator requires a *feedback resistor to reduce DC gain and prevent saturation.*
  - The op amp operates within the *linear range* so that it does not saturate.

VI. Applications

VI.1. Integrator

Ex 6.9: Find v_O in the op amp circuit if $v_1 = 10.\cos 2t$ *(mV)* and $v_2 = 0,5t$ *(mV)* *(assume that the voltage across the capacitor is initially zero)*



- This is a summing integrator

$$v_0 = -\frac{1}{R_1C}\int v_1 dt - \frac{1}{R_2C}\int v_2 dt$$

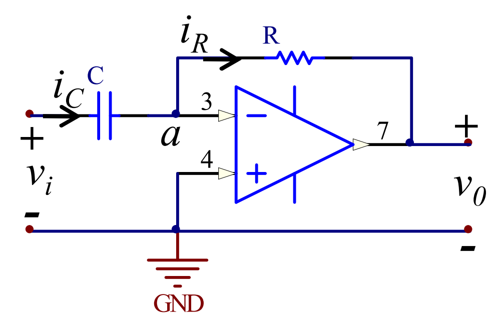
$$v_0 = -\frac{1}{3.10^6.2.10^{-6}}\int_0^t 10\cos 2t dt - \frac{1}{100.10^3.2.10^{-6}}\int_0^t 0,5t dt$$

$$v_0 = -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0,2}\frac{0,5t^2}{2} = -0,833\sin 2t - 1,25t^2\,(mV)$$

VI. Applications

VI.2. Differentiator

- An *differentiator* is an op amp circuit whose output is proportional to the rate of change of the inputs signal.



- Applying KCL at node *a*

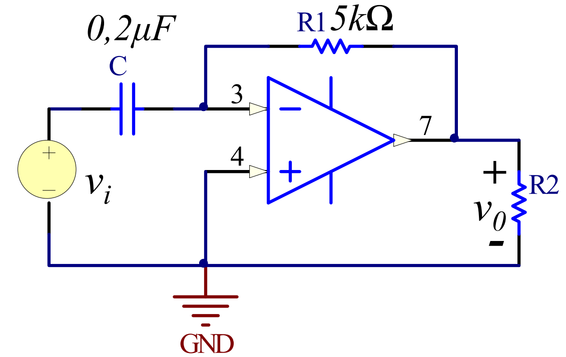
$$i_R = i_C \leftrightarrow -\frac{v_0}{R} = i_R = i_C = C\frac{dv_i}{dt}$$

$$\boxed{v_0(t) = -RC\frac{dv_i}{dt}}$$

- Note that:
  - Differentiator circuits are *electronically unstable* because any electrical noise within the circuit is exaggerated by the differentiator.
  - Differentiator circuit is not as useful and popular as the integrator.

VI. Applications

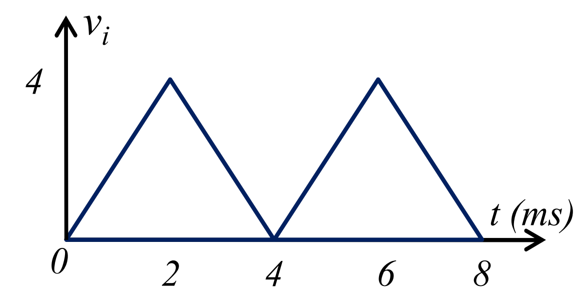
VI.2. Differentiator



Ex 6.10: Find the output voltage with the given input. Take $v_O = 0$ at $t = 0$.

- This is a differentiator with: $RC = 5.10^3.0,2.10^{-6} = 10^{-3}$
- For 0 < t < 4ms or 4 < t < 8ms, the input voltage is:

$$v_i = \begin{cases} 2t & 0 < t < 2ms, 4 < t < 6ms \\ 8 - 2t & 2 < t < 4ms, 6 < t < 8ms \end{cases}$$



$$\rightarrow v_O = -RC\frac{dv_i}{dt} = \begin{cases} -2mV & 0 < t < 2ms, 4 < t < 6ms \\ 2mV & 2 < t < 4ms, 6 < t < 8ms \end{cases}$$